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On the Fixed Homogeneous Circle Problem

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Abstract

We give some results about the dynamics of a particle moving in Euclidean three-space under the influence of the gravitational force induced by a fixed homogeneous circle. Our main results concern (1) singularities and (2) the dynamics in the plane that contains the circle. The study presented here is purely analytic.

In this paper we present a study about the movement of a particle in Euclidean three-space $\mathbb{R}^3$ on which the only acting force is the gravitational force induced by a fixed homogeneous circle. Even though this problem seems quite natural, we could not find in the literature any reference concerning the dynamics of it. All we could find were a few different ways of expressing the potential function. Essentially all these expressions had already appeared in Poincare's Théorie du Potentiel Newtonien [4], published first in 1899. Probably the most well-known expressions are the one expressed in terms of elliptic integrals of the first kind and the one using the arithmetic-geometric mean given by Gauss. We believe this is the first study presented about the dynamics of this problem.

Before we present our first result we need some notation and a couple of definitions. We want to study the movement in $\mathbb{R}^3$ of a particle P under the influence of the gravitational force induced by a fixed homogeneous circle $\mathcal{C}$. Denote by $\mathbf{r} = (x, y, z) \in \mathbb{R}^3 - \mathcal{C}$ the position of the particle P. Also $\dot{\mathbf{r}} = (\dot{x}, \dot{y}, \dot{z})$ denotes its velocity. According to Newton's Law the movement of P obeys the following second order differential equation:

$$\ddot{\mathbf{r}} = -\nabla V(\mathbf{r}) \tag{0.1}$$

where V denotes the potential energy induced by $\mathcal{C}$. The expression of V is given by $V(\mathbf{r}) = -\int_{\mathcal{C}} \frac{\lambda\, du}{\|\mathbf{r}-u\|}$, where λ is the constant mass density of the circle $\mathcal{C}$. We say that a solution $\mathbf{r}(t)$, defined in the maximal interval (a, b), has a singularity at b (or at a), if $b < +\infty$ (if $a > -\infty$). For $p \in \mathbb{R}^3$, let $dist(p, \mathcal{C})$ denote the Euclidean distance, $inf_{x\in\mathcal{C}}\|p - x\|$, from p to $\mathcal{C}$. We can now ask: if $lim_{t\to b^-} dist(\mathbf{r}(t), \mathcal{C}) = 0,\ b < +\infty$, does $\mathbf{r}(t)$ approach a well defined point in $\mathcal{C}$? A priori, $\mathbf{r}(t)$ could approach the circle without getting closer to any specific point in the circle. A singularity at c is called a *collision singularity* if there exists $\mathbf{r}^* \in \mathcal{C}$ such that $\mathbf{r}(t) \to \mathbf{r}^*$, when $t \to c$. Otherwise, the singularity is called a singularity without collision. Here is our first result.

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Theorem A. *All singularities in the fixed homogeneous circle problem are collision singularities.*

In what follows we consider the fixed homogeneous circle $\mathcal{C}$ contained in the xy-plane and centered at the origin. Also, by rescaling we can consider the circle with radius equal to one (see section 1.3). Then the mass M of $\mathcal{C}$ is given by $M = 2\pi\lambda$. A quick examination of the problem shows that (see section 1.2 for more details) the *z-axis*, the *horizontal plane* (i.e. the xy-plane, which contains the circle) and any *vertical plane* (i.e any plane that contains the z-axis) are invariant subspaces of our problem. Also, any *radial line* (i.e. any line in the xy-plane that passes through the origin) is an invariant subspace. Here by an invariant subspace Λ we mean that any solution that begins tangentially in Λ is totally contained in Λ. The restriction of the problem to the z-axis is a one dimensional problem which is not difficult to treat. In fact it is a special case of a problem studied by Sitnikov (see [2], [6]). Our two next results discuss the dynamics restricted to the horizontal plane. This problem is a central force problem. For $r \in \mathbb{R} - \{1\}$ define the function $V(r) = V(r\mathbf{u})$, where $\mathbf{u}$ is any vector in the xy-plane of length one (note that we are using the same letter V for two different functions). The dynamics restricted to the horizontal plane has two cases: inside the circle and outside the circle. It is a classical result that using the polar coordinates (r, θ) of a point (x, y) in the horizontal plane it can be proved that system 0.1 in this case is equivalent to: $\ddot{r} = \frac{K^2}{r^3} - \frac{d}{dr}V(r)$, $\dot{\theta} = \frac{K}{r^2}$ (see [3]). Here the constant K is the angular momentum of the solution. Hence this problem can be reduced in a canonical way to a problem with one degree of freedom (the variable $r(t)$) and once we know $r(t)$, we can obtain $\theta(t)$ by integration. Note that $K = 0$ if and only if the solution lies in a radial line (i.e. $\theta(t)$ is constant). Equation $\ddot{r} = \frac{K^2}{r^3} - \frac{d}{dr}V(r)$ is equivalent to

$$\ddot{r}(t) = -\frac{d}{dr}U(r(t)), \tag{0.2}$$

where U denotes the *effective potential* $U(r) = \frac{K^2}{2r^2} + V(r)$. Note that $U(r)$ depends on K, that is for each K we may have different functions $U(r)$ (maybe we should write $U_K(r)$ but we do not want to complicate our notation). Hence to determine the dynamics we have to know the behavior of the effective potential for all values of K. Note that outside the circle the effective potential is defined for $1 < r < \infty$ and inside the circle the effective potential is defined for $0 < r < 1$ (for $K \neq 0$) and for $-1 < r < 1$ (for $K = 0$).

Theorem B. *The effective potential $U(r)$ inside the circle has the following properties:*

i. $\frac{d}{dr}V(r) < 0$, *for* $0 < r < 1$. *Hence* $\frac{d}{dr}U(r) < 0$, *for* $0 < r < 1$.

ii. $lim_{r\to 1^-} U(r) = -\infty$.

iii. *For* $K \neq 0$ *we have* $\lim_{r\to 0^+} U(r) = \infty$.

iv. *For* $K = 0$ *we have that* $U(r)$, $-1 < r < 1$, *is even.*

The following are direct consequences of the Theorem above.

1. The graphs of $U(r)$ for the cases $K = 0$ and $K \neq 0$ have the following form.

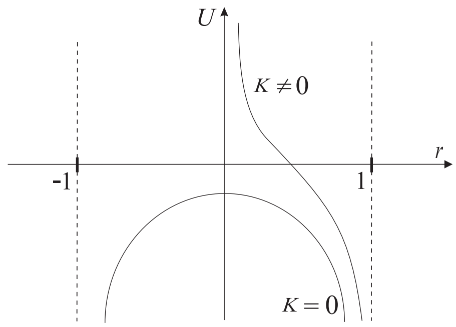


Figure 0.1: Graph of the effective potential U inside the circle.

Note that for $K = 0$, $U(r)$, $-1 < r < 1$, has an absolute maximum value $E_0 = U(0) = V(0) = -M = -2\pi\lambda$, at $r = 0$. Hence $r \equiv 0$ or, equivalently $\mathbf{r}(t) = 0$, $t \in \mathbb{R}$, is an equilibrium solution.

2. By (i) of the Theorem, the force inside the circle is repulsive.

3. As mentioned before, for $K = 0$ a solution $\mathbf{r}(t)$ (with polar coordinates $(r(t), \theta(t))$ stays in a radial line, i.e. θ is constant. Without loss of generality we can assume that this radial line is the x-axis, that is $\theta = 0$. Hence $\mathbf{r}(t) = (x(t), 0, 0)$, where $x(t) = r(t)$. Let $E = E(x, \dot{x})$ denote the energy of $\mathbf{r}(t) = (x(t), 0, 0)$. It follows that the phase portrait for $K = 0$ (i.e the level curves of $E = E(x, \dot{x})$ in the $x\dot{x}$-plane) has the following form:

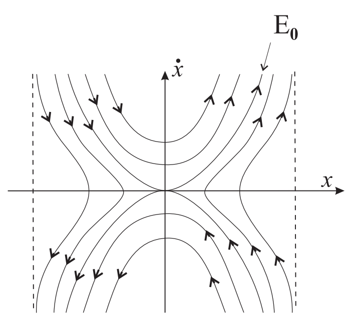
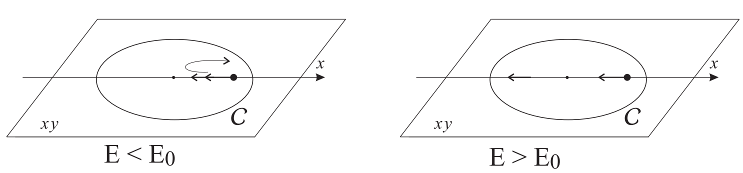


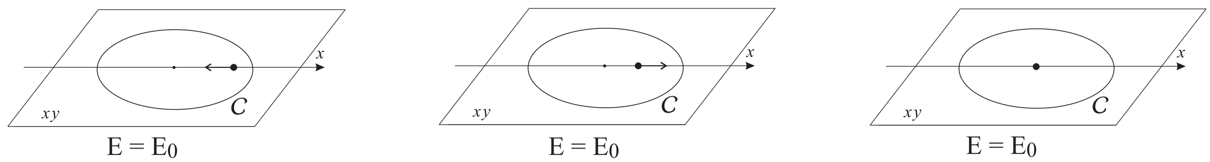
Figure 0.2: Phase portrait for $K = 0$.

Consequently, depending on the energy the particle behaves in the following way. If $E < E_0$ the particle comes from the point $p = (\pm 1, 0, 0)$ in the circle, stops before reaching the origin and then turns back to the point p. If $E > E_0$ the particle comes from the point p in the circle, and goes all the way to the point $-p$.



If $E = E_0$ we have three cases: the particle comes from the point p in the circle and converges to the origin (in infinite time), or comes from the origin (in infinite time) and converges to p, or

stays at rest in the origin.



4. For $K \neq 0$ we have that a solution $\mathbf{r}(t)$ never passes through the origin. By Theorem B, this solution comes from the circle and returns to it (in finite time, by Theorem A). Note that there is a unique t_0 such that $\mathbf{r}(t_0)$ is the closest point to the origin. Since $\dot{r}(t_0) = 0$ we have that $\dot{\mathbf{r}}(t_0)$ is perpendicular to $\mathbf{r}(t_0)$. We say that $\mathbf{r}(t)$ is *normalized* if $\mathbf{r}(t_0)$ lies in the positive y-axis. Note that, by symmetry, any solution (inside the circle) with $K \neq 0$ can be obtained from a normalized one by a rotation. (See Fig. 0.3 below.)

The following two Propositions complement the results given by Theorem B, for the case of a solution inside the circle.

Proposition 1. *Let $\mathbf{r}(t)$ be a solution of 0.1, contained in the horizontal plane and inside the circle. Let (a, b) be its maximal interval of definition. If $\mathbf{r}(t)$ approaches the circle as $t \to b^-$ or $t \to a^+$, then*

(1) *$b < \infty$ or $-\infty < a$, respectively. Hence, by Theorem A, the solution collides to a point in the circle,*

(2) *when $\mathbf{r}(t)$ converges to the circle the speed converges to infinity and the velocity $\dot{\mathbf{r}}(t)$ becomes orthogonal to the circle.*

It follows from this Proposition that the intervals of definition (a, b) of solutions inside the circle are as follows: (1) for $K \neq 0$ we have that a and b are both finite (2) for $K = 0$ and $E \neq E_0$ we also have a and b are finite (3) for $K = 0$ and $E = E_0$ either $a = -\infty$ and $b = \infty$ or exactly one of a and b is finite.

Proposition 2. *Let $\mathbf{r}(t)$ be a solution of 0.1, contained in the horizontal plane, inside the circle and with angular momentum $K \neq 0$. If $\mathbf{r}(t)$ is normalized then the trace of the curve $\mathbf{r}(t)$ is the graph of an even convex function $y = y(x)$.*

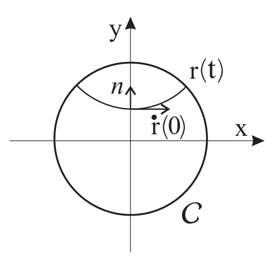


Figure 0.3: Trace of the normalized solution $\mathbf{r}(t)$.

We now state the results concerning the effective potential $U(r)$ for the case $r > 1$, that is outside the circle.

Theorem C. *The effective potential* $U(r)$, $r > 1$ *(outside the circle) has the following properties:*

i. $\frac{d}{dr}V(r) > 0$. *Hence* $\frac{d}{dr}(U(r) - \frac{K^2}{2r^2}) > 0$.

ii. $lim_{r\to 1^+} U(r) = -\infty$.

iii. $lim_{r\to\infty} U(r) = 0$.

Also there are $K_0 > 0$, r_0, $1 < r_0 < 2$, *and continuous functions* $r_1 = r_1(|K|)$, $r_2 = r_2(|K|)$ *defined for* $|K| \geq K_0$, *with* $1 < r_1 \leq r_0 \leq r_2$ *such that:*

iv. *For* $K \in (-K_0, K_0)$, $U(r)$ *does not have critical points. Hence from ii.* $\frac{d}{dr}U(r) > 0$ *and* $U(r)$ *is an increasing function.*

v. *For* $K \notin (-K_0, K_0)$ *the critical points of* $U(r)$ *are exactly* $r_1 = r_1(|K|)$, $r_2 = r_2(|K|)$.

vi. $\frac{d}{dr}U(r) > 0$ *for* $r < r_1$ *and* $r > r_2$. *Also* $\frac{d}{dr}U(r) < 0$ *for* $r_1 < r < r_2$.

vii. $r_1(K_0) = r_2(K_0) = r_0$.

viii. r_1 *is a decreasing function and* $lim_{K\to\infty} r_1 = 1$.

ix. r_2 *is an increasing function and* $lim_{K\to\infty} r_2 = \infty$.

x. $lim_{K\to\infty} U(r_1(K)) = \infty$ *and* $lim_{K\to\infty} U(r_2(K)) = 0$.

Remark. The special value r_0 does not depend on the density λ (or on the mass M) of the circle (see the proof of Theorem C). Since $K_0^2 = r_0^3 \frac{d}{dr}V(r_0)$, the value K_0 does depend on λ. It is interesting to compare this with Corollary 1.8.

It follows from Theorem C that the graphs of $U(r)$ have the following form:

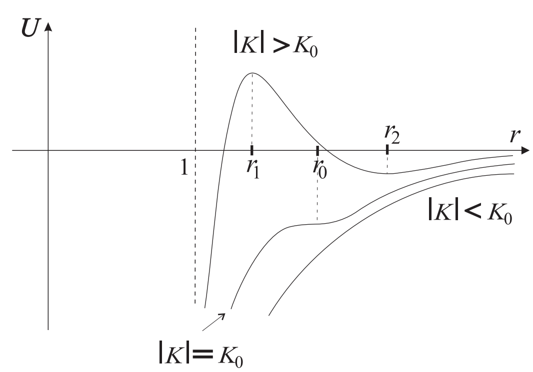


Figure 0.4: Graph of the effective potential U outside the circle.

Note that for $K \notin [-K_0, K_0]$ the function $U(r)$ has a local maximum at r_1 and a local minimum at r_2. By $(x.)$ of Theorem C, the local maximum value $U(r_1)$ at r_1 is eventually (as $K \to \infty$ and $r_1 \to 1^-$) a global maximum and $U(r_1) \to +\infty$. The following are direct consequences of Theorem C.

1. From (i) of the Theorem, the force outside the circle is attractive.

2. We discuss the stability of circular solutions. Here by stability we mean the following. A circular solution $\mathbf{r}(t)$ in the horizontal plane is *stable* if its reduced solution $r(t) = \|\mathbf{r}(t)\| =$ constant is a *stable* equilibrium position of the reduced problem $\ddot{r}(t) = -\frac{d}{dr}U(r(t))$. It is easy to see that a circular solution $\mathbf{r}(t)$ is stable iff the trace of solutions with initial conditions close to the initial conditions of $\mathbf{r}(t)$ stay close to the trace of $\mathbf{r}(t)$. Recall that for a fixed K, a circular solution with radius r has momentum K if and only if r is a critical point of $U(r)$. Hence for $K \in (-K_0, K_0)$ there are no circular solutions with momentum K. For every $K \notin [-K_0, K_0]$ there are exactly two circular solutions with radii $r_1(|K|)$ and $r_2(|K|)$. The circular solution with radius r_1 is not stable and there are "spiral" solutions approaching it from the inside and the outside. The circular solution with radius r_2 is stable and has no "spiral" solutions approaching it. For $K = K_0$ there is exactly one circular solution and it has radius r_0. This circular solution is unique among all circular solutions. It is not stable and have spiral solutions approaching it only from the inside. Also it can be approximated by stable circular orbits from the outside (with varying momenta K). It follows that a circular solution with radius r is stable if and only is $r > r_0$. Note that a non-circular solution with momentum K_0 either collides to the circle, escapes to infinity or is the solution that approaches the circle (in infinite time).

3. Let $E = E(r, \dot{r})$ denote the energy of $r(t) = \|\mathbf{r}(t)\|$. Using Theorem C we can deduce the phase portraits. We have three cases: $K \in (-K_0, K_0)$, $K = K_0$, $K \notin [-K_0, K_0]$. For the first two cases the phase portraits are shown in figures 0.5 and 0.6 respectively, and the solutions have the following behavior:

(i) $K \in (-K_0, K_0)$. In this case all solutions either collide to the circle or escape to infinity.

(ii) $K = K_0$. In this case all solutions (different from the unique circular solution of radius r_0) either collide to the circle or escape to infinity or converge to the circular solution with radius r_0 (in infinite time).

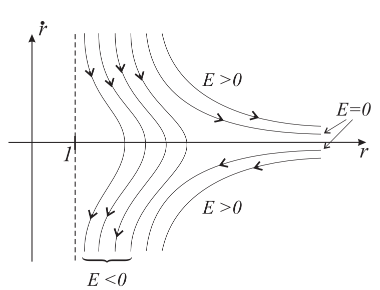


Figure 0.5: $K \in (-K_0, K_0)$.

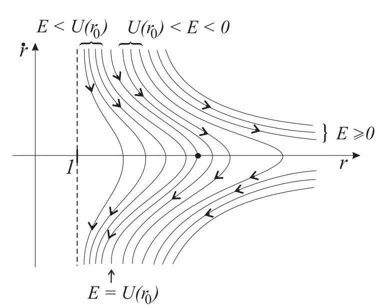


Figure 0.6: $K = K_0$

(iii) We analise now the case $K \notin [-K_0, K_0]$. Let $\overline{E} = U(r_1)$. We have also three cases: $\overline{E} > 0, \overline{E} < 0, \overline{E} = 0$ (see figures 0.7, 0.8, 0.9). Note that, by (*vi.*), (*vii.*) and (*x.*) of Theorem C all three cases do happen. For $\overline{E} > 0$ we have (see figure 0.7): In region I the solutions are bounded, and converge to the circle. In region II the solutions are unbounded, come from infinity and converge to the circle. In region III the solutions are unbounded, come from the circle and escape to infinity. In region IV the solutions are unbounded, approach the circular solution (with radius r_1) and escape to infinity. In region V the solutions are bounded, and stay close to the circular solution (with radius r_2).

In figure 0.7 the solution represented by γ_1 comes from the circle and converges to the unstable circular solution (with radius r_1). The solution represented by γ_2 comes from the circular solution (with radius r_1) and converges to the circle. The solution represented by γ_3 comes from the circular solution (with radius r_1) and escapes to infinity. The solution represented by γ_4 comes from infinity and converges to the circular solution (with radius r_1).

For $\overline{E} = 0$ we have also three cases: $E < \overline{E}$, $E = \overline{E}$, $E > \overline{E}$. We have the following analysis (see figure 0.8 below): For $E < \overline{E}$ in region I the solutions are bounded, and converge to the circle; in region IV the solutions are bounded and stay close to the circular solution (with radius r_2). For $E > \overline{E}$ we have: In region II the solutions are unbounded, come from the circle and escape to infinity. In region III the solutions are unbounded, come from infinity and converge to the circle. For $E = \overline{E}$ we have: The solution represented by γ_1 comes from the circle and converges to the unstable circular solution (with radius r_1). The solution represented by γ_2 comes from the circular solution (with radius r_1) and converges to the circle. The solution represented by γ_3 comes from the circular solution (with radius r_1) and escapes to infinity. The solution represented by γ_4 comes from infinity and converges to the circular solution (with radius r_1).

For $\overline{E} < 0$ we have four cases: $E < \overline{E}$, $E \geq 0$, $\overline{E} < E < 0$, $E = \overline{E}$ (see figure 0.9 below). For the case $E < \overline{E}$ in region I the solutions are bounded and converge to the circle; in region IV the solutions are bounded and stay close to the circular solution (with radius r_2). For $E \geq 0$ we have: In region II the solutions are unbounded, come from the circle and escape to infinity. In region III the solutions are unbounded, come from infinity and converge to the circle. For $\overline{E} < E < 0$ the solutions are bounded, come from the circle, approaching the circular solutions and return converging to the circle. Finally, for $E = \overline{E}$ we have: The solution represented by γ_1 comes from the circle and converges to the unstable circular solution (with radius r_1). The solution represented by γ_2 comes from the circular solution (with radius r_1) and converges to the circle. The solution represented by γ_3 comes from the circular solution with radius r_1, approach the circular solution with radius r_2, and return converging to the circular solution with radius r_1.

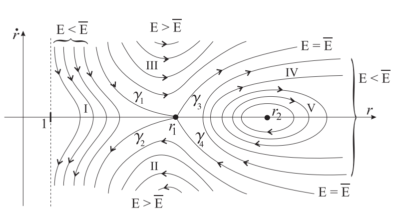


Figure 0.7: $\overline{E} = U(r_1) > 0$.

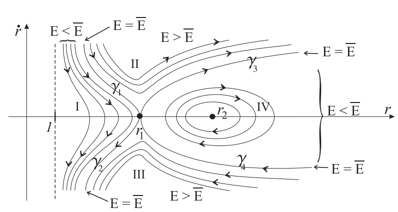


Figure 0.8: $\overline{E} = U(r_1) = 0$.

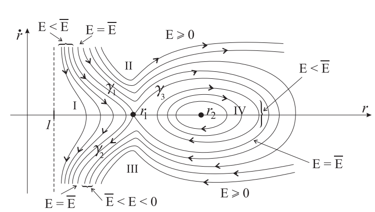


Figure 0.9: $\overline{E} = U(r_1) < 0$.

We mentioned above that a circular solution of radius r in the horizontal plane is stable (among solutions in the horizontal plane) if and only if $r > r_0$. The following question arises: are these solutions stable among all solutions in $\mathbb{R}^3$? The answer is affirmative, but before we state this result we have some comments. Since our original problem (in $\mathbb{R}^3$) is invariant by rotations around the z-axis, the problem can be reduced in a canonical way to a problem with two degrees of freedom (see section 5). In fact, every solution $\mathbf{r}(t) = (x(t), y(t), z(t))$ can be written as $\mathbf{r}(t) = (r(t)cos\varphi(t), r(t)sin\varphi(t), z(t))$, i.e the cylindrical coordinates of $\mathbf{r}(t)$ are $(r(t), \varphi(t), z(t))$, and $(r(t), \varphi(t), z(t))$ satisfy a system of equations (system (5.7)). Moreover, once we know $r(t)$ we can obtain $\varphi(t)$ by integration. We say that $(r(t), z(t))$ is the *canonical projection solution* of $\mathbf{r}(t)$. That is, the canonical projection solution is obtained from the solution by the map $(x, y, z) \mapsto (r, z)$, $r = \sqrt{x^2 + y^2}$. We have that $(r(t), z(t))$ satisfy the first two equations of (5.7). Note that under this projection the circle $\mathcal{C}$ projects to a point $x_{\mathcal{C}}$ with rz coordinates (1,0). We extend the definition of a circular solution (for central force problems) to this context in the following way: We say that a solution is *circular* if its canonical projection solution $(r(t), z(t))$ is an equilibrium position of the system formed by the first two equations of (5.7). Also, we say that a circular solution is *stable* if its canonical projection solution is a *stable* equilibrium position of the system formed by the first two equations of (5.7). It is easy to see that a circular solution $\mathbf{r}(t)$ is stable iff the trace of solutions with initial conditions close to the initial conditions of $\mathbf{r}(t)$ stay close to the trace of $\mathbf{r}(t)$.

Proposition 3. *All circular solutions lie in the horizontal plane. Moreover, a circular solution (in the horizontal plane) of radius r is stable in $\mathbb{R}^3$ if and only if $r > r_0$, where r_0 is as in Theorem C.*

We present two more results. Fix $\lambda > 0$. For $\epsilon > 0$ denote by $W(\mathbf{r}, \epsilon)$ the potential function induced by a the fixed homogeneous circle in the xy-plane centered at $(\frac{1}{\epsilon}, 0, 0)$ with radius $\frac{1}{\epsilon}$ and density λ. We write $W(x, z; \epsilon)$ for the restriction of this potential to the xz-plane. Define $\nabla W(x, z; 0) := 64\,\lambda \frac{(x,z)}{x^2+z^2}$, $(x, z) \neq (0, 0)$. Let $\nabla W(x, z; \epsilon)$ be the gradient of $W(x, z; \epsilon)$ with respect to the variables x and z. Also, let $A = \{\,(x, z; \epsilon)\,;\,(0, 0) \neq (x, y) \neq (\frac{2}{\epsilon}, 0)\,\}$. Thus $\nabla W(x, z; \epsilon)$ is defined on A.

Proposition 4. ∇W *is continuous on A.*

Note that $\nabla W(x, z; 0) = 64\lambda \nabla ln(\sqrt{x^2 + z^2})$ and recall that $2\lambda\, ln(\sqrt{x^2 + z^2}) = \lambda ln(x^2 + z^2)$ is the potential induced by the infinite wire (with constant density λ and infinite mass) orthogonal to xz-plane intersecting the xz-plane at the origin. Hence by Proposition 4 the problem of the fixed homogeneous circle with large radius, constant density λ, and centered at $(\frac{1}{\epsilon}, 0, 0))$ can be regarded as a perturbation of the problem of the infinite homogeneous straight wire with density 32λ. (To obtain the same density λ, instead of 32λ, it is enough to take circles with radius $\frac{1}{2\epsilon}$.) This result is a key element in the proof of the existence of periodic orbits near the circle (see [1]).

There are a few textbooks where it is claimed that the potential function of fixed homogeneous circles with large radii "converge" to the potential function of the infinite straight wire, but we have not found satisfactory proofs of these claims. In fact, it seems that in order to avoid

the problems with the infinities, we have to work with the gradients, instead of the potentials. This necessity seems to be related to the fact that if we try to calculate the potential of the infinite straight wire by integrating $\int_{-\infty}^{+\infty} \frac{dx}{\|p-(x,0,0)\|}$ we obtain infinity, but the integral for the gradient does converge (or can be computed easily from Gauss formula).

Finally we present our last result. The potential V of our problem at P can be written as $V(P) = -\frac{M}{\sigma(D,d)}$, where $\sigma(D,d)$ is the arithmetic-geometric mean of $D > 0$ and $d > 0$ (see Remark 1.1). Here d, D are the maximum and minimum distances $D = D(P)$, $d = d(P)$ of the point P to the circle. As we mentioned above, this formula of $V(P)$ was given by Gauss. In our last result we give a formula for $(\frac{\partial}{\partial D}V, \frac{\partial}{\partial d}V)$ in terms of D, d and their successive geometric and arithmetic means d_j, D_j.

Proposition 5. *We have the following formulas:*

$$\frac{\partial}{\partial D}V(D,d) = \frac{\chi - 1}{D}V(D,d), \quad \frac{\partial}{\partial d}V(D,d) = -\frac{\chi}{d}V(D,d),$$

where $\chi = \sum_{n=1}^{+\infty} \frac{1}{2^n}\frac{d_{n-1}}{D_n}\prod_{j=1}^{n-1}\left(1 - \frac{d_{j-1}}{D_j}\right)$.

This paper has 7 sections. In section 1 we give some preliminary results that will be needed later. These include some basic facts about symmetry, elementary properties of the potential function as well as some identities related to the potential function. In sections 2, 3 and 5 we prove Theorems A, B and C, respectively. At the end of section 5 we prove Proposition 3 and also prove that the only equilibrium position is the origin. By symmetry it is clear that the origin is an equilibrium position but it is not too trivial to prove that it is the only one. In section 4 we prove Propositions 1 and 2. Finally in sections 6 and 7 we prove Propositions 4 and 5, respectively.

1 Preliminaries.

1.1 The Potential.

As before consider the fixed homogeneous circle $\mathcal{C}$ with constant density λ and radius ρ, contained in the xy-plane, and centered at the origin. The mass M of $\mathcal{C}$ is $2\pi\lambda\rho$. We denote by $P = (x, y, z)$ the coordinates of the position of a particle in $\mathbb{R}^3 \setminus \mathcal{C}$ and let $V(P)$ be the potential at P induced by $\mathcal{C}$. Let $D = D(P)$ and $d = d(P)$ denote the maximum and minimum distances from the particle P to the circle. We have the following expressions for these distances: $D^2 = (\sqrt{x^2+y^2}+\rho)^2 + z^2$ and $d^2 = (\sqrt{x^2+y^2}-\rho)^2 + z^2$. Also we have that (see [7], p.196)

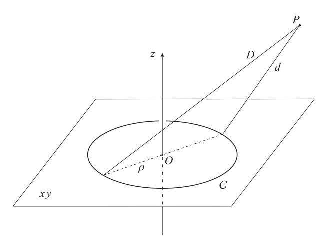


Figure 1.10: Fixed circle centered at the origin.

$$V(P) = -4\lambda\rho\int_0^{\frac{\pi}{2}} \frac{d\psi}{\sqrt{d^2cos^2\psi + D^2sin^2\psi}}. \tag{1.3}$$

Define $T : \{(x,y);\, x>0, y>0\} \longrightarrow \mathbb{R}$ by $T(d,D) = \int_0^{\frac{\pi}{2}} \frac{d\psi}{\sqrt{d^2cos^2\psi + D^2sin^2\psi}}$. Then we have $V(P) = -4\lambda\rho T(d,D)$. By the change of variable $\psi = \frac{\pi}{2} - \theta$, it can be proved that T is symmetric with respect to d and D, that is, $T(d,D) = T(D,d)$. It was proved by Gauss (see [7], p.197-199) that $V(P) = -\frac{M}{\sigma(d,D)}$ where $\sigma(d,D)$ is the arithmetic-geometric mean of d and D. Hence $T(d,D) = \frac{\pi}{2\sigma(d,D)}$.

Remark 1.1 Recall that the arithmetic-geometric mean $\sigma(m,n)$ of two positive numbers m,n is defined in the following way. Set $m_1 = \frac{m+n}{2}$ and $n_1 = \sqrt{mn}$ and define inductively the sequences $\{m_i\}$, $\{n_i\}$ by $m_{i+1} = \frac{m_i+n_i}{2}$, $n_{i+1} = \sqrt{m_in_i}$. We have $m > m_1 > m_2 > ... > n_2 > n_1 > n$. It can be shown that $\lim m_i = \lim n_i$. This common limit is called the *arithmetic-geometric mean* $\sigma(m,n)$ of m and n. Note that if $m=n$, then $\sigma(m,m) = m$. Note also that $\sigma(m,n) = \sigma(m_i,n_i)$. Since $T(m,n) = \frac{\pi}{2\sigma(m,n)}$, we also have $T(m,n) = T(m_i,n_i)$.

Define now $f(t) = \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{cos^2\theta + tsin^2\theta}}$, $\quad 0 < t \le 1$. Then $T(d,D) = \dfrac{f\left(\frac{d^2}{D^2}\right)}{D} = \dfrac{\pi}{2\sigma(d,D)}$, $0 < d \le D$. Hence $V(P) = -\frac{4\lambda\rho}{D} f\left(\frac{d^2}{D^2}\right)$. It follows that $f\left(\frac{d^2}{D^2}\right) = \frac{\pi D}{2\sigma(d,D)}$, $\quad 0 < d \le D$. Setting $D=1$ we have that $f(d^2) = \frac{\pi}{2\sigma(d,1)}$, $\quad d \le 1$.

Remarks.

(1) $f(t) = \displaystyle\int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{cos^2\theta + tsin^2\theta}} = \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1-(1-t)sin^2\theta}} = K(\sqrt{1-t}\,)$, where K is the elliptic integral of the first kind.

(2) $f(t) = \frac{\pi}{2}\left(1 + \displaystyle\sum_{n=1}^{+\infty} \left(\frac{1\cdot 3\cdots(2n-1)}{2\cdot 4\cdots 2n}\right)^2 (1-t)^n\right)$. This series can be obtained from the power series of K (see [7], p.197).

Lemma 1.2 *(i) f is a decreasing function and f' is an increasing function.*
(ii) $\sigma(d,1) \ge \sqrt{d}$
(iii) $f(t) \le \frac{\pi}{2}\frac{1}{\sqrt[4]{t}}$
(iv) $|f'(t)| \le \frac{1}{2}\frac{f(t)}{t}$
(v) $|f'(t)| \le \frac{\pi}{4}\frac{1}{t^{5/4}}$
(vi) $\lim_{t\to 0^+} f(t) = +\infty$
(vii) $\lim_{t\to 0^+} tf'(t) = -\frac{1}{2}$

Proof. (i) $f'(t) = -\frac{1}{2}\int_0^{\frac{\pi}{2}} \frac{sin^2\theta d\theta}{(cos^2\theta + tsin^2\theta)^{3/2}} < 0$, which implies that f is a decreasing function. $f''(t) = \frac{3}{4}\int_0^{\frac{\pi}{2}} \frac{sin^4\theta d\theta}{(cos^2\theta + tsin^2\theta)^{5/2}} > 0$, which implies that f' is an increasing function. Note that $f^{(n)}(t) < 0$, if n is odd, and $f^{(n)}(t) > 0$, if n is even. (ii) By the definition of σ, we have that $\sigma(d,1) \ge$ (geometric mean of d and 1) $= \sqrt{d}$. (iii) Setting $D=1$, $d^2 = t$, and applying (ii), we obtain (iii). (iv) $|f'(t)| = \frac{1}{2}\int_0^{\frac{\pi}{2}} \frac{sin^2\theta d\theta}{(cos^2\theta + tsin^2\theta)^{3/2}} = \frac{1}{2}\int_0^{\frac{\pi}{2}} \frac{1}{t}\frac{1}{(cos^2\theta + tsin^2\theta)^{1/2}}\frac{tsin^2\theta}{(cos^2\theta + tsin^2\theta)}d\theta \le \frac{1}{2}\frac{f(t)}{t}$, (because $\frac{tsin^2\theta}{(cos^2\theta + tsin^2\theta)} < 1$, $\ t > 0$.) The proof of (v) follows directly from (iii) and (iv). (vi) Since $\sqrt{cos^2\theta + tsin^2\theta} \le \sqrt{cos^2\theta + t} \le cos\theta + \sqrt{t}$ we have $f(t) \ge \int_0^{\frac{\pi}{2}} \frac{d\theta}{cos\theta + \sqrt{t}}$. A

change of variable shows $\int_0^{\frac{\pi}{2}} \frac{d\theta}{cos\theta+\sqrt{t}} = \int_0^{\frac{\pi}{2}} \frac{d\theta}{sin\theta+\sqrt{t}}$. Hence, since $sin\theta \leq \theta$ we have $f(t) \geq \int_0^{\frac{\pi}{2}} \frac{d\theta}{sin\theta+\sqrt{t}} \geq \int_0^{\frac{\pi}{2}} \frac{d\theta}{\theta+\sqrt{t}} = ln(\frac{\pi}{2}+\sqrt{t})-ln(\sqrt{t}) \to \infty$ as $t \to 0^+$. (vii) We have the following power series for the elliptic integral of the first kind $K(k)$ (see [7], p.203), $K(k) = \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1-k^2 sin^2\theta}} = \left[1+\frac{k_1^2}{4}+\frac{9}{64}k_1^4+\ldots\right] ln\frac{4}{k_1} - \left[\frac{k_1^2}{4}+\frac{21}{128}k_1^4+\ldots\right]$, where $k_1^2 = 1-k^2$. Since $f(t) = K(\sqrt{1-t})$, taking $k_1^2 = t$ we have $f(t) = \left[1+\frac{t}{4}+\frac{9}{64}t^2+\ldots\right] ln\frac{4}{\sqrt{t}} - \left[\frac{t}{4}+\frac{21}{128}t^2+\ldots\right]$. It follows that $f'(t) = \left[\frac{1}{4}+\frac{18}{64}t+\ldots\right] ln\frac{4}{\sqrt{t}} - \left[\frac{1}{4}+\frac{42}{128}t+\ldots\right] + \left[1+\frac{t}{4}+\frac{9}{64}t^2+\ldots\right]\left(-\frac{1}{2}\frac{1}{t}\right)$ therefore $lim_{t\to 0}f'(t)t = -\frac{1}{2}$. ■

1.2 Symmetries of the potential.

In this section we study the symmetries of the fixed homogeneous circle problem and determine the invariant subspaces. As before, we are considering the fixed homogeneous circle $\mathcal{C}$, contained in the xy-plane and centered at the origin. We call the xy-plane (which contains $\mathcal{C}$) by *horizontal plane*, and any plane that contains the z-axis by *vertical plane*.

A *symmetry* ψ of our problem is a transformation $\psi : \mathbb{R}^3 - \mathcal{C} \to \mathbb{R}^3 - \mathcal{C}$ that preserve V, i.e V is invariant by ψ, that is $V\psi = V$. The symmetries of V are evident from the geometry of the problem: they are the rigid transformations of $\mathbb{R}^3$ that preserve $\mathcal{C}$:

Proposition 1.3 *The potential V is invariant by:*
(i) Rotations about the z-axis,
(ii) Reflection with respect to the horizontal plane,
(iii) Reflections with respect to a vertical plane.

Proof. It follows from (1.3) and from the expressions of D and d (see section 1.1). ■

Corollary 1.4 *The gradient field ∇V is invariant by:*
(i) Rotations about the z-axis,
(ii) Reflection with respect to the horizontal plane,
(iii) Reflections with respect to a vertical plane.

Proof. Since all symmetries above are isometries of $\mathbb{R}^3 \setminus \mathcal{C}$ (with the canonical flat metric), the Corollary follows from the Proposition above and from a classical result in the elementary theory of mechanical systems on Riemannian manifolds. ■

Corollary 1.5 *If $\mathbf{r}(t)$ is a solution of (0.1), then $\psi\,\mathbf{r}(t)$ is also a solution of (0.1), where ψ is one of the symmetries above.* ■

Corollary 1.6 *The z-axis, the horizontal plane and the vertical planes are invariant subspaces, that is, a solution of (0.1) that starts in one of these spaces, with velocity contained in it, stays there for all the time in which the solution is defined.*

Proof. It follows from the following facts:

- the z-axis is the invariant space of all the symmetries in (i) of Corollary 1.4,
- the horizontal plane is the invariant space of the reflection in (ii) of Corollary 1.4,
- a vertical plane is the invariant space of a refection in (iii) of Corollary 1.4. ■

Note that a *radial line* (i.e. any line in the xy-plane that passes through the origin) is also an invariant subspace since it is the intersection of the horizontal plane with a vertical plane.

1.3 Properties.

We will need to consider circles with variable radius and mass. Write $V(\mathbf{r},\rho,M)$ to denote the potential induced by $\mathcal{C}$, contained in the xy-plane and centered at the origin, with radius ρ and mass M, and $\nabla V(\mathbf{r},\rho,M)$ to denote the gradient (with respect to $\mathbf{r}$) of $V(\mathbf{r},\rho,M)$.

Lemma 1.7 *The potential V of the fixed homogeneous circle problem satisfies the following identities:*
(i) $V(\mathbf{r},\rho,c\,M) = c\,V(\mathbf{r},\rho,M)$, for $c \in I\!R$,
(ii) $V(c\,\mathbf{r},c\,\rho,M) = \frac{1}{c}\,V(\mathbf{r},\rho,M)$, for $c > 0$,
(iii) $\nabla V(c\,\mathbf{r},c\,\rho,M) = \frac{1}{c^2}\,\nabla V(\mathbf{r},\rho,M)$, for $c > 0$,
(vi) $\nabla V(\mathbf{r},\rho,c\,M) = c\,\nabla V(\mathbf{r},\rho,M)$, for $c \in I\!R$.

Proof. It follows directly from the definition of $V(\mathbf{r},\rho,M) = -\frac{M}{2\pi}\int_0^{2\pi}\frac{d\theta}{\|\mathbf{r}-\rho e^{i\theta}\|}$, where $e^{i\theta} = (cos\theta, sin\theta, 0)$. ■

Corollary 1.8 *Let ρ,ζ,M,N be positive numbers. If $\mathbf{r}(t)$ is a solution of $\ddot{\mathbf{r}}\,(t) = -\nabla V(\mathbf{r},\rho,M)$ then $\mathbf{s}(t) = \frac{\zeta}{\rho}\mathbf{r}\left(\sqrt{\frac{N\rho^3}{M\zeta^3}}\;t\right)$ is a solution of $\ddot{\mathbf{s}}\,(t) = -\nabla V(\mathbf{s},\zeta,N)$.*

Proof. It follows from Lemma 1.7, by a direct calculation. ■

Remark 1.9 Note that if $\mathbf{r}(t)$ and $\mathbf{s}(t)$ are as above, then they have the same qualitative properties. For instance, if $\mathbf{r}(t)$ is periodic then $\mathbf{s}(t)$ is also periodic (with period $\frac{T\sqrt{M\zeta^3}}{\sqrt{N\rho^3}}$, where T is the period of $\mathbf{r}(t)$).

Note that Lemma 1.7 and Corollary 1.8 imply that in the study the fixed homogeneous circle problem we can assume the mass and the radius to be equal to one.

2 Singularities: Proof of Theorem A.

To prove Theorem A we first show the following Proposition, which is of general nature. In the next Proposition *dist* denotes "Euclidean distance".

Proposition 2.1 *Let $V : \Omega \to I\!R$, $\Omega \subseteq I\!R^n$ open. Let $\mathbf{r}(t)$, $t \in (a,b)$, be a solution of $\ddot{\mathbf{r}} = -\nabla V(\mathbf{r})$ such that there exist:*
(1) $v_0, v_1 \in I\!R$, $v_0 < v_1$, with $dist(V^{-1}(v_0), V^{-1}(v_1)) > 0$,

(2) $t_1 < s_1 < t_2 < s_2 \ldots, \quad t_i, s_i \in (a,b), \ \ i \in \mathbb{N}$*, satisfying the following two properties: (a)* $V(\mathbf{r}(t_i)) = v_0$*, and (b)* $V(\mathbf{r}(s_i)) = v_1$*. Then* $b = +\infty$*.*

Proof. Since V is continuous, for every interval $[t_i, s_i]$ we can choose, for all i, an interval $[t_i^*, s_i^*] \subseteq [t_i, s_i]$ such that $v_0 = V(\mathbf{r}(t_i^*)) \leq V(\mathbf{r}(t)) \leq V(\mathbf{r}(s_i^*)) = v_1$, for all $t \in [t_i^*, s_i^*]$.

Since the total energy $E = E(\mathbf{r}(t)) = V(\mathbf{r}(t)) + \frac{1}{2}\|\dot{\mathbf{r}}(t)\|^2$ is constant, we have that $\|\dot{\mathbf{r}}(t)\| = \sqrt{2(E-V)}$, with $V = V(\mathbf{r}(t))$ and E constant. Then we have $\|\dot{\mathbf{r}}(t)\| \leq \sqrt{2(E-v_0)}$, for $t \in [t_i^*, s_i^*]$. Since the length of the curve $\mathbf{r}(t)$ between $[t_i^*, s_i^*]$ is larger than the distance $d = dist(V^{-1}(v_0), V^{-1}(v_1)) > 0$, we have

$$d \leq \int_{t_i^*}^{s_i^*} \|\dot{\mathbf{r}}(t)\| dt \leq \int_{t_i^*}^{s_i^*} \sqrt{2(E-v_0)} dt = (s_i^* - t_i^*)\sqrt{2(E-v_0)} \leq (s_i - t_i)\sqrt{2(E-v_0)}.$$

Hence, $\frac{d}{\sqrt{2(E-v_0)}} \leq (s_i - t_i)$, for all i. Since we have an infinite number of disjoint intervals (t_i, s_i), with $0 < \frac{d}{\sqrt{2(E-v_0)}} \leq (s_i - t_i)$ for all i, we conclude that $b = +\infty$. ■

In what follows of this section let $\mathcal{C}$ be the fixed homogeneous circle in $\mathbb{R}^3$ centered at the origin, contained in the horizontal plane and with constant density λ and radius 1. Also let V be the potential induced by $\mathcal{C}$ and $\mathbf{r}(t)$ be a solution of $\ddot{\mathbf{r}} = -\nabla V(\mathbf{r})$, defined in the maximal interval (a,b).

Proposition 2.2 *If* $b < +\infty$ *then* $lim_{t \to b^-} dist(\mathbf{r}(t), \mathcal{C}) = 0$*. Analogously, if* $a > -\infty$ *then* $lim_{t \to a^+} dist(\mathbf{r}(t), \mathcal{C}) = 0$*.*

To prove this Proposition we need the following Lemmas. Let $\{\mathbf{r}_n\}_{n \in \mathbb{N}} \subset \mathbb{R}^3 - \mathcal{C}$.

Lemma 2.3 $lim_{n \to +\infty} V(\mathbf{r}_n) = 0$ *if and only if* $lim_{n \to +\infty} \|\mathbf{r}_n\| = +\infty$*.*

Proof. Suppose first that $\|\mathbf{r}_n\| \to +\infty$. For all $u \in \mathcal{C}$, we have $\|u\| = \rho$. Hence, for $\|\mathbf{r}\| > \rho$, and $u \in \mathcal{C}$, we have $\|\mathbf{r} - u\| \geq |\|\mathbf{r}\| - \|u\|| = |\|\mathbf{r}\| - \rho| \geq \|\mathbf{r}\| - \rho$. Hence $0 \leq -V(\mathbf{r}) = \int_{\mathcal{C}} \frac{\lambda}{\|\mathbf{r}-u\|} du \leq \int_{\mathcal{C}} \frac{\lambda}{\|\mathbf{r}\|-\rho} du = \frac{M}{\|\mathbf{r}\|-\rho}$, where $M = \int_{\mathcal{C}} \lambda \, du$ is the mass of $\mathcal{C}$. It follows that $lim_{n \to +\infty} V(\mathbf{r}_n) = 0$.

Suppose now that $V(\mathbf{r}_n) \to 0$. Since $\|u\| = \rho$, for $u \in \mathcal{C}$, we have $\frac{1}{\|\mathbf{r}-u\|} \geq \frac{1}{\|\mathbf{r}\|+\rho}$. Consequently, $0 \leq \frac{M}{\|\mathbf{r}\|+\rho} = \int_{\mathcal{C}} \frac{\lambda}{\|\mathbf{r}\|+\rho} du \leq \int_{\mathcal{C}} \frac{\lambda}{\|\mathbf{r}-u\|} du = -V(\mathbf{r})$. Therefore, $lim_{n \to +\infty} \|\mathbf{r}_n\| = +\infty$. This proves the Lemma. ■

Lemma 2.4 $lim_{n \to +\infty} V(\mathbf{r}_n) = -\infty$ *if and only if* $lim_{n \to +\infty} dist(\mathbf{r}_n, \mathcal{C}) = 0$*.*

Proof. Suppose that $lim_{n \to +\infty} V(\mathbf{r}_n) = -\infty$. Since $0 < dist(\mathbf{r}_n, \mathcal{C}) \leq \|\mathbf{r}_n - u\|$, for all $u \in \mathcal{C}$, we have $0 < -V(\mathbf{r}_n) = \int_{\mathcal{C}} \frac{\lambda}{\|\mathbf{r}_n - u\|} du \leq \int_{\mathcal{C}} \frac{\lambda}{dist(\mathbf{r}_n, \mathcal{C})} du = M \frac{1}{dist(\mathbf{r}_n, \mathcal{C})}$. It follows that $dist(\mathbf{r}_n, \mathcal{C}) \to 0$.

Conversely, suppose that $dist(\mathbf{r}_n, \mathcal{C}) \to 0$. Let d, D be as in section 1.1. We have $d(\mathbf{r}_n) = dist(\mathbf{r}_n, \mathcal{C}) \to 0$ and $D(\mathbf{r}_n) \to 2\rho$. It follows from (vi) of Lemma 1.2 that $V(\mathbf{r}_n) = \frac{-4\lambda}{D(\mathbf{r}_n)} f(\frac{d^2(\mathbf{r}_n)}{D^2(\mathbf{r}_n)}) \to -\infty$. ■

Proof of Proposition 2.2. First note that, by the two Lemmas above, $V^{-1}(c)$ is compact, for all $c < 0$. Now, let $\mathbf{r}(t)$, $t \in (a,b)$, $b < +\infty$, be a maximal solution of $\ddot{\mathbf{r}} = -\nabla V(\mathbf{r})$.

Then $lim_{t\to b^-}V(\mathbf{r}(t))$ exists (it could be finite or infinite), otherwise, by the continuity of V we could choose two sequences (t_n), (s_n) such that $t_1 < s_1 < t_2 < s_2 < \ldots,\ t_i, s_i \in (a,b)$, with $V(\mathbf{r}(t_i)) = v_0, V(\mathbf{r}(s_i)) = v_1$, $v_0 < v_1$. Since $V^{-1}(v_0)$ and $V^{-1}(v_1)$ are disjoint compact and non-empty, we have that $dist(V^{-1}(v_0), V^{-1}(v_1)) > 0$. Hence, by Proposition 2.1 we would have that $b = +\infty$, a contradiction.

Since $image\, V \subset (-\infty, 0)$, it follows that $lim_{t\to b^-}V(\mathbf{r}(t))$ is either 0, or it is a number $v^* \neq 0$, or it is equal to $-\infty$. We show that the two first possibilities do not happen. It will then follow that $lim_{t\to b^-}V(\mathbf{r}(t)) = -\infty$ and therefore, by Lemma 2.4, we would have that $lim_{t\to b^-}dist(\mathbf{r}(t), \mathcal{C}) = 0$, which proves the Proposition.

If $lim_{t\to b^-}V(\mathbf{r}(t)) = 0$, then there exists t_0 such that for all $t > t_0$, $v_0 < V(\mathbf{r}(t)) < 0$, with $v_0 = V(\mathbf{r}(t_0))$. On the other hand by Lemma 2.3, we have that $\|\mathbf{r}(t)\| \to +\infty$ when $t \to b^-$. Moreover, since $E = V(\mathbf{r}(t)) + \frac{1}{2}\|\dot{\mathbf{r}}(t)\|^2$, E is constant then $\|\dot{\mathbf{r}}(t)\| < \sqrt{2(E - v_0)}$, for all $t \in (t_0, b)$, that is, the velocity is bounded in this interval. We also have that $\int_{t_0}^{t}\|\dot{\mathbf{r}}(t)\|dt \geq \|\mathbf{r}(t) - \mathbf{r}(t_0)\| \geq dist(\mathbf{r}(t), V^{-1}(v_0)) =: d_t$. Hence, $d_t \leq (t - t_0)\sqrt{2(E - v_0)}$, and then, $t_0 + \frac{d_t}{\sqrt{2(E-v_0)}} \leq t$, for all $t \in (t_0, b)$.

Since $V^{-1}(v_0)$ is compact, there exists $r > 0$ such that $V^{-1}(v_0) \subset B(0, r)$ and since $\|\mathbf{r}(t)\| \to +\infty$, given $n > 0$, there exists t_n such that for all $t \in [t_n, b)$, $\mathbf{r}(t) \notin B(0, n)$. This implies that, $d_t = dist(\mathbf{r}(t), V^{-1}(v_0)) > n - r$, for all $b > t \geq t_n$. In particular, $d_{t_n} \geq n - r$, and for $b > t_n > t_0$, we have $b \geq t_n \geq t_0 + \frac{n-r}{\sqrt{2(E-v_0)}}$. Hence, $lim_{n\to\infty}\left(t_0 + \frac{n-r}{\sqrt{2(E-v_0)}}\right) = +\infty$, and we conclude that $b = +\infty$, a contradiction.

Finally suppose that $lim_{t\to b^-}V(\mathbf{r}(t)) = v^*$, with $v^* \in (-\infty, 0)$. Hence, given $\varepsilon > 0$, there exists t_ε such that for all $t > t_\varepsilon$, $V(\mathbf{r}(t)) \in (v^* - \varepsilon, v^* + \varepsilon)$. Let $v_0 = v^* - \varepsilon$, $v_1 = v^* + \varepsilon$; we can suppose that $0 < \varepsilon < |v^*|$, hence $v_1 < 0$. Clearly the set $V^{-1}([v_0, v_1])$ is not empty and since V is continuous, this set is closed in $\mathbb{R}^3 \setminus \mathcal{C}$. Moreover, by Lemma 2.4, this set is closed in $\mathbb{R}^3$, and by Lemma 2.3 it is bounded (because $v_1 < 0$). Hence, $V^{-1}([v_0, v_1])$ is compact and $\mathbf{r}(t) \in V^{-1}([v_0, v_1])$, for all $t > t_\varepsilon$.

On the other hand, from the energy equation, we have $\|\dot{\mathbf{r}}(t)\|^2 = 2(E - V(\mathbf{r}(t)))$, therefore, for all $t > t_\epsilon$, we have $\|\dot{\mathbf{r}}(t)\| \leq \sqrt{2(E - v_0)} = c_2$. Hence the maximal solution $(\mathbf{r}(t), \dot{\mathbf{r}}(t))$, $t \in (a, b)$, of the system of first order differential equations

$$\left\{\begin{array}{lcl} \dot{\mathbf{r}} & = & \mathbf{v} \\ \dot{\mathbf{v}} & = & -\nabla V(\mathbf{r}) \end{array}\right.$$

is contained in the compact $V^{-1}([v_0, v_1]) \times \overline{B(0, c_2)}$, for $t \in (t_\epsilon, b)$. It follows from a classical result in the elementary theory of differential equations (see [5]) that $b = +\infty$, a contradiction. This proves the Proposition. ■

Proof of Theorem A. Let $\mathbf{r}(t)$, $t \in (a, b)$, $b < +\infty$, be a maximal solution of $\ddot{\mathbf{r}} = -\nabla V(\mathbf{r})$. By Proposition 2.2, we have that $lim_{t\to b^-}dist(\mathbf{r}(t), \mathcal{C}) = 0$. To prove that $lim_{t\to b^-}\mathbf{r}(t) = \mathbf{r}^* \in \mathcal{C}$, we write this system in cylindrical coordinates (r, θ, z) in $(\mathbb{R}^3 \setminus \{z - \text{axis}\})$. We have (see equations (5.7) of section 5):

$$\ddot{r} = \frac{K^2}{r^3} - \frac{\partial V}{\partial r}, \qquad \ddot{z} = -\frac{\partial V}{\partial z},$$

with $\dot{\theta} = \dfrac{K}{r(t)^2}$, where K is constant and $V(r,z) = V(r\,cos\theta,\, r\,sin\theta, z)$.

Since the circle in cylindrical coordinates is given by $\mathcal{C} = \{(\rho, \varphi, 0), \varphi \in \mathbb{R}\}$, we have that showing $lim_{t\to b^-}\mathbf{r}(t) = \mathbf{r}^* \in \mathcal{C}$ is equivalent to showing $lim_{t\to b^-} r(t) = \rho$, $lim_{t\to b^-} z(t) = 0$ and $lim_{t\to b^-}\theta(t) = \theta_0$, for some θ_0. The two first limits follow from the fact that $lim_{t\to b^-} dist(\mathbf{r}(t), \mathcal{C}) = 0$. We will now prove that $lim_{t\to b^-}\theta(t) = \theta_0$. If $K = 0$, $\theta(t)$ is constant, and we have nothing to prove. Suppose then that $K > 0$. Hence $\theta(t)$ is an increasing function. Thus, to prove that the limit of $\theta(t)$ exists, it is enough to prove that $\theta(t)$ is bounded above, for t in a neighborhood of b.

Since $r(t) \to \rho$, there exists t_0 such that, for $t > t_0$, $r(t) > \frac{\rho}{2}$. Hence $\theta(t) = \displaystyle\int_{t_0}^{t} \frac{K}{r(s)^2} ds + \theta(t_0) \leq \int_{t_0}^{t} \frac{4K}{\rho^2} ds + \theta(t_0) \leq \frac{4K(b - t_0)}{\rho^2} + \theta(t_0) < +\infty$ for all $t \in (t_0, b)$. Therefore, the limit of $\theta(t)$ when $t \to b^-$ exists. This proves the Theorem. ■

3 The Dynamics Inside the Circle: Proof of Theorem B.

In this section we consider again the fixed homogeneous circle $\mathcal{C}$ contained in the xy-plane, centered at the origin and with radius 1 and constant density λ. Let $\mathbf{r} = (x, y)$ be the position of the particle in this plane, under the influence of the gravitational attraction induced by $\mathcal{C}$. Let $r = \|\mathbf{r}\|$ and θ be the polar coordinates of $\mathbf{r} = (x, y)$. Also, let D and d be as in section 1.1. In the horizontal plane we have $D^2 = (r + 1)^2$ and $d^2 = (r - 1)^2 = (1 - r)^2$. Hence the expression for the potential (1.3) becomes

$$V(\mathbf{r}) = V(r) = -4\lambda \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{(r+1)^2 cos^2\theta + (r-1)^2 sin^2\theta}}$$

Since V depends only on r, we have a central force problem and (see [3], ch.3) the system of equations $\ddot{\mathbf{r}} = -\nabla V(\mathbf{r})$ (restricted to the horizontal plane) is equivalent to the system:

$$\begin{cases} \ddot{r} = \dfrac{K^2}{r^3} - \dfrac{d}{dr} V(r) \\[2ex] \dot{\theta} = \dfrac{K}{r^2} \end{cases} \tag{3.4}$$

where $K = \langle\, \mathbf{r} \times \dot{\mathbf{r}},\, e_3 \,\rangle = x\dot{y} - \dot{x}y$ is the angular momentum. Note that $\ddot{r} = \frac{K^2}{r^3} - \frac{d}{dr}V(r) = \frac{d}{dr}\left(\frac{-K^2}{2r^2} - V(r)\right)$ which is equivalent to

$$\ddot{r} = -\frac{d}{dr} U(r), \tag{3.5}$$

where $U(r) = \dfrac{K^2}{2r^2} + V(r)$. The function $U(r)$ is called the *effective potential* .

Proposition 3.1 *For* $0 < r < 1$, $\frac{d}{dr}V(r) < 0$.

Proof. In the horizontal plane and inside the circle we have $D = r + 1$ and $d = 1 - r$. Hence $D_1 = 1$ and $d_1 = \sqrt{1 - r^2}$, where D_1 and d_1 are the arithmetic and geometric means of D and d, respectively. Therefore, by Remark 1.1 of section 1.1, we have

$$V(\mathbf{r}) = V(r) = -4\lambda T(1 + r, 1 - r) = -4\lambda T(1, \sqrt{1 - r^2}) = -4\lambda \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1 - r^2 sin^2\theta}}. \tag{3.6}$$

Differentiating we have $\frac{d}{dr}V(r) = -4\lambda \int_0^{\frac{\pi}{2}} \frac{r\, sin^2\theta}{\left(1 - r^2 sin^2\theta\right)^{3/2}}\, d\theta \ < \ 0.$ ■

Proof of Theorem B. Differentiating we have $\frac{d}{dr}U(r) = -\frac{K^2}{r^3} + \frac{d}{dr}V(r)$. Hence, by the Proposition above we have $\frac{d}{dr}U(r) < 0$ for $0 < r < 1$. This proves (i) of Theorem B.

When $r \to 1^-$ we have that the particle tends to the circle. Hence, by Lemma 2.4, $V(r)$ tends to $-\infty$. Consequently $U(r) = \frac{K^2}{2r^2} + V(r)$ tends also to $-\infty$. This proves (ii). Note that (iii) and (iv) follow from the definition of $U(r)$ and (3.6). ■

4 Proofs of Propositions 1 and 2.

First we prove Proposition 2, but before we give some remarks. Let $\mathbf{r}(t)$ be as in the statement of Proposition 2. Without loss of generality we can assume that $t_0 = 0$ (see definition of a normalized solution).

(1) Recall that the curvature of the curve $\mathbf{r}(t)$ is by definition

$$k(t) = \frac{1}{\|\dot{\mathbf{r}}(t)\|^2}\langle \ddot{\mathbf{r}}(t), \mathbf{n}(t)\rangle \ = \ \frac{1}{\|\dot{\mathbf{r}}(t)\|^3}\langle \ddot{\mathbf{r}}(t),\, \|\dot{\mathbf{r}}(t)\|\,\mathbf{n}(t)\rangle \ = \ \frac{1}{\|\dot{\mathbf{r}}(t)\|^3}\langle \ddot{\mathbf{r}}(t), \dot{\mathbf{r}}^{\perp}(t)\rangle,$$

where $\mathbf{n}(t) = \frac{(-\dot{y}(t), \dot{x}(t))}{\|(-\dot{y}(t), \dot{x}(t))\|}$ and $\dot{\mathbf{r}}^{\perp}(t) = \|\dot{\mathbf{r}}(t)\|\,\mathbf{n}(t) = (-\dot{y}(t), \dot{x}(t))$.

(2) Note that $k(t)$ is continuous and is never zero because if $k(t) = 0$ for some t, then $\ddot{\mathbf{r}}(t) \perp \mathbf{n}(t)$ and follows that $\dot{\mathbf{r}}(t)$ is radial (because $\ddot{\mathbf{r}}(t)$ is always radial), and then the solution $\mathbf{r}(t)$ would be radial, that is $K = 0$, a contradiction.

(3) $\dot{\mathbf{r}}(0) \perp \ddot{\mathbf{r}}(0)$ and $\ddot{\mathbf{r}}(0)$ points upward. It follows that $\mathbf{n}(0) = c\ddot{\mathbf{r}}(0)$, for some $c > 0$. Then $k(0) = \bar{c}\,\|\ddot{\mathbf{r}}(0)\|^2 > 0$, with $\bar{c} > 0$. In this case, since k is continuous and is always non-zero, we have that $k > 0$, for all t.

Proof of Proposition 2. Let $\mathbf{r}(t) = (x(t), y(t))$ be as in the statement of Proposition 2 and recall that we are assuming $t_0 = 0$. Let (a, b) be the maximal interval on which $\mathbf{r}(t)$ is defined. Then $a < 0 < b$.

Claim. $\dot{x}(t) \neq 0$, for all $t \in (a, b)$.

We prove that $\dot{x}(t) \neq 0$, for all $t \in [0, b)$, the proof for $t \in (a, 0]$ is similar. Suppose that there exists $t_0 \in [0, b)$ such that $\dot{x}(t_0) = 0$. Let $t_0 = min\{t \geq 0\,;\, \dot{x}(t) = 0\}$. Since $\dot{x}$ is continuous and $\dot{x}(0) > 0$, it follows that $t_0 > 0$. Then $\dot{x}(t) > 0$, for $t \in [0, t_0)$. We have $\dot{\mathbf{r}}(t_0) = (\dot{x}(t_0), \dot{y}(t_0)) = (0, \dot{y}(t_0))$, with $\dot{y}(t_0) \neq 0$, because $\dot{\mathbf{r}}(t) \neq 0$, for all t. We can write $\dot{\mathbf{r}}(t) = a(t)(cos\varphi(t), sin\varphi(t))$, with $a(t) = \|\dot{\mathbf{r}}(t)\| > 0$ and $\varphi(0) = 0$, φ continuous. Differentiating: $\ddot{\mathbf{r}}(t) = \dot{a}(t)(cos\varphi(t), sin\varphi(t)) + a(t)\dot{\varphi}(t)(-sin\varphi(t), cos\varphi(t))$. Hence $k(t) = \frac{1}{\|\dot{\mathbf{r}}(t)\|^3}\langle\, \ddot{\mathbf{r}}(t), \dot{\mathbf{r}}^{\perp}(t)\,\rangle = \frac{a^2(t)\dot{\varphi}(t)}{a^3(t)} = \frac{\dot{\varphi}(t)}{a(t)}$. Because the curvature is positive (see Remark (3) above) this shows that $\dot{\varphi}(t) > 0$ and $\varphi(t)$ is an increasing function. Since $\dot{x}(t_0) = 0$, $\varphi(t_0) = \frac{\pi}{2}$ (hence $\dot{y}(t_0) > 0$), or $\varphi(t_0) = \frac{3\pi}{2}$ (hence $\dot{y}(t_0) < 0$). Since φ is an increasing function we have that $\varphi(t_0) = \frac{\pi}{2}$, which implies that $\dot{y}(t_0) > 0$. Since x is increasing on $(0, t_0)$, we have that $x(t_0) > 0$. Recall that $\ddot{\mathbf{r}}(t)$ is radial and expansive, that is, $\ddot{\mathbf{r}}(t) = b(t)\mathbf{r}(t) = b(t)(x(t), y(t))$, with $b(t) > 0$. Hence $k(t_0) = b(t_0)\,\langle\,(x(t_0), y(t_0)), (-\dot{y}(t_0), 0)\,\rangle = b(t_0)(-x(t_0)\dot{y}(t_0)) < 0$, a contradiction. Therefore there is no t_0 such that $\dot{x}(t_0) = 0$. This proves the claim. ■

It follows from the claim that $\dot{x}(t) > 0$, $a < t < b$. Hence $x(t)$ is a increasing function. In this way the function $t \to x(t)$ is one-to-one and it follows that $x(t)$ possesses an inverse $t = t(x)$. Define $f(x) = y(t(x))$. Note that the graph of f is equal to the trace of $\mathbf{r}$. Differentiating f with respect to x, we have

$$\frac{d}{dx}f(x) = \frac{d}{dt}y(t(x))\frac{d}{dx}t(x) = \frac{\frac{d}{dt}y(t(x))}{\frac{d}{dt}x(t(x))}.$$

Differentiating again we have

$$\frac{d^2}{dx^2}f(x) = \frac{1}{\dot{x}^3}\,\langle\, \ddot{\mathbf{r}}, \|\dot{\mathbf{r}}\|\,\mathbf{n}\,\rangle\ = \frac{k\|\dot{\mathbf{r}}\|^3}{\dot{x}^3} > 0$$

because $\dot{x} > 0$ and $k > 0$. Hence the trace of $\mathbf{r}(t)$ is given by the graph of a convex function f. By the symmetry of the problem, the solution $\mathbf{r}(t)$ is symmetric with respect to the y-axis. Hence the function f is even. ■

Proof of Proposition 1. Let $\mathbf{r}(t)$ and (a, b) be as in the statement of Proposition 1. Let $r(t) = ||\mathbf{r}(t)||$. Then $r(t)$ satisfies $\ddot{r} = -\frac{d}{dr}U(r)$. Suppose that $\mathbf{r}(t) \to 1^-$, when $t \to b^-$. By Theorem B we have $lim_{t\to b^-}U(r(t)) = lim_{r\to 1^-}U(r) = -\infty$. Thus there exists t_0 such that $U(r(t)) < E - 1$, for $t \in [\,t_0, b)$, where $E = \frac{1}{2}\dot{r}^2 + U(r)$ is the energy of $r(t)$. Hence, for $t \geq t_0$, $\dot{r}(t) \neq 0$. Moreover, $\dot{r}(t) > 0$ (because $r(t) \to 1^-$). Consequently, $r(t)$ is one-to-one on $[\,t_0, b)$, and has an inverse $t = t(r)$, $r_0 \leq r < 1$, $r_0 = r(t_0)$. From $E = \frac{1}{2}\dot{r}^2 + U(r)$, we have that $\dot{r} = \sqrt{2(E - U(r))}$. Then $\int_{t_0}^{t(r)} dt = \int_{r_0}^{r}\frac{dr}{\sqrt{2(E-U(r))}}$ and it follows that $t(r) = \int_{r_0}^{r}\frac{dr}{\sqrt{2(E-U(r))}} + t_0$, $r_0 \leq r < 1$. In this way $b = lim_{r\to 1^-}t(r) = \int_{r_0}^{1}\frac{dr}{\sqrt{2(E-U(r))}} + t_0 < \int_{r_0}^{1}\frac{dr}{\sqrt{2}} + t_0 \leq \frac{1}{\sqrt{2}}(1 - r_0) + t_0 < +\infty$. This proves part (1) of Proposition 1.

We now prove part (2) of Proposition 1. Again, let $\mathbf{r}(t) = (x(t), y(t))$ and (a, b) be as in the statement of Proposition 1. Assume $\|\mathbf{r}(t)\| \to 1$, when $t \to b^-$. We have $E = \frac{1}{2}\|\dot{\mathbf{r}}\|^2 + V(\mathbf{r})$.

Then $\|\dot{\mathbf{r}}\|^2 = 2(E - V(\mathbf{r}))$. When $t \to b^-$, $\|\mathbf{r}\| \to 1$ hence $V(\mathbf{r}) \to -\infty$. Therefore $\|\dot{\mathbf{r}}\| \to +\infty$. Without loss of generality suppose that $\mathbf{r}(t)$ converges to the point $(1,0)$ of the circle, that is $x(t) \to 1$ and $y(t) \to 0$ when $t \to b^-$.

We shall show that $\dot{\mathbf{r}}(t)$ becomes horizontal as $t \to b^-$, that is $\frac{\dot{y}(t)}{\dot{x}(t)} \to 0$ when $t \to b^-$. First, note that $\dot{x}(t) \neq 0$ for t close to b. Moreover, $\dot{x}(t) \to +\infty$ when $t \to b^-$. To see this suppose that there exists a sequence $t_n \to b^-$ with $|\dot{x}(t_n)| < M$ for some M. Then $|\dot{y}(t_n)| \to +\infty$ (because $\|\dot{\mathbf{r}}(t_n)\| \to +\infty$). Hence $|K| = |\dot{x}(t_n)y(t_n) - \dot{y}(t_n)x(t_n)| \to +\infty$, a contradiction because K is constant.

Now, suppose that $lim_{t\to b^-}\frac{|\dot{y}(t)|}{|\dot{x}(t)|} \neq 0$. This implies that there exists a sequence $\{t_n\}$ with $t_n \to b^-$ such that $\frac{|\dot{y}(t_n)|}{|\dot{x}(t_n)|} \geq \delta$, for some $\delta > 0$. Hence, we have $|K| = |x(t_n)\dot{y}(t_n) - y(t_n)\dot{x}(t_n)| \geq |x(t_n)||\dot{y}(t_n)| - |y(t_n)||\dot{x}(t_n)| \geq |x(t_n)|(\delta|\dot{x}(t_n)|) - |y(t_n)||\dot{x}(t_n)| = |\dot{x}(t_n)|\Big(\delta|x(t_n)| - |y(t_n)|\Big)$. Taking the limit when $t_n \to b^-$, we have that $|K| \to +\infty$, a contradiction. Therefore, $lim_{t\to b}\frac{|\dot{y}(t)|}{|\dot{x}(t)|} = 0$. ■

5 Dynamics Outside the Circle: Proof of Theorem C.

As in section 3 we consider the fixed homogeneous circle $\mathcal{C}$ contained in the xy-plane, centered at the origin and with radius 1 and constant density λ. Also $\mathbf{r}$ will denote the position of a particle in this plane, under the influence of the gravitational attraction induced by $\mathcal{C}$. Let $r = \|\mathbf{r}\|$.

Proof of Theorem C. Let $D = D(\mathbf{r})$ and $d = d(\mathbf{r})$ be as in section 1.1, that is they are the maximum and minimum distances from $\mathbf{r}$ to the circle. In the horizontal plane and outside the circle we have $D = r + 1$ and $d = r - 1$. Hence $D_1 = r$ and $d_1 = \sqrt{r^2 - 1}$, where D_1 and d_1 are the arithmetic and geometric means of D and d, respectively. Therefore, by Remark 1.1 of section 1.1, we have

$$V(\mathbf{r}) = V(r) = -4\lambda T(r+1, r-1) = -4\lambda T(r, \sqrt{r^2-1}) = -4\lambda \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{r^2 cos^2\theta + (r^2-1)sin^2\theta}} = -4\lambda \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{r^2 - sin^2\theta}}$$

Differentiating we have

$$\frac{d}{dr}V(r) = 4\lambda \int_0^{\frac{\pi}{2}} \frac{r}{\Big(r^2 - sin^2\theta\Big)^{3/2}}\, d\theta \ > \ 0.$$

This proves (i) of Theorem C. When $r \to 1^+$ we have that the particle tends to the circle. Hence, by Lemma 2.4, $V(r)$ tends to $-\infty$. Consequently $U(r) = \dfrac{K^2}{2r^2} + V(r)$ tends also to $-\infty$. This proves (ii). Note that (iii) follows from the definition of $U(r)$ and Lemma 2.3.

We now prove (iv)-(ix) of the statement of Theorem C. Since by definition $U(r) = \frac{K^2}{2r^2} + V(r)$ we have that $\frac{d}{dr}U(r) = 0$ if and only if $\frac{K^2}{r^3} = \frac{d}{dr}V(r)$, or equivalently $K^2 = r^3\frac{d}{dr}V(r)$. Define $g(r) = r^3\frac{d}{dr}V(r),\ r > 1$. By the formula above we have $g(r) = 4\lambda \displaystyle\int_0^{\frac{\pi}{2}} \frac{r^4}{\Big(r^2 - sin^2\theta\Big)^{3/2}}\, d\theta.$

Lemma 5.1 *The function $g(r)$ has the following properties*

a. $lim_{r \to 1^-} g(r) = \infty$,

b. $\lim_{r \to \infty} \frac{g(r)}{r} = 2\pi\lambda$,

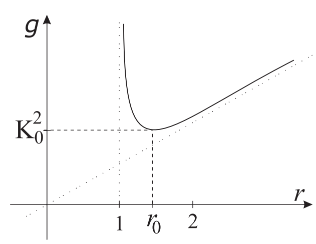
c. *$g(r)$ has exactly one critical point r_0, $1 < r_0 < 2$.*

Proof. If $r \to 1^-$ the particle approaches the circle therefore, by Lemma 2.4 $V(r) \to -\infty$. Hence $\frac{d}{dr}V(r)$ is not bounded when $r \to 1^-$. But $\frac{d^2}{dr^2}V(r) - 4\lambda \int_0^{\frac{\pi}{2}} \frac{2r^2 + sin^2\theta}{(r^2 - sin^2\theta)^{5/2}}\, d\theta < 0$ therefore $\frac{d}{dr}V(r) \to \infty$, as $r \to 1^-$. This proves $(a.)$. For $(b.)$ we have $\frac{g(r)}{r} = 4\lambda \int_0^{\frac{\pi}{2}} \frac{r^3}{(r^2 - sin^2\theta)^{3/2}}\, d\theta = 4\lambda \int_0^{\frac{\pi}{2}} \frac{1}{(1 - \frac{sin^2\theta}{r^2})^{3/2}}\, d\theta \to 4\lambda \frac{\pi}{2} = 2\pi\lambda$. To prove $(c.)$ we compute the first two derivatives of $g(r)$. A direct calculation shows:

$$\tfrac{d}{dr}\, g(r) \;=\; 4\lambda\, r^3 \int_0^{\frac{\pi}{2}} \frac{r^2 - 4sin^2\theta}{\left(r^2 - sin^2\theta\right)^{5/2}}\, d\theta, \qquad \tfrac{d^2}{dr^2}\, g(r) \;=\; 4\lambda\, r^2 \int_0^{\frac{\pi}{2}} \frac{3r^2\, sin^2\theta + 12sin^4\theta}{\left(r^2 - sin^2\theta\right)^{7/2}}\, d\theta.$$

It follows that $\frac{d^2}{dr^2}\, g(r) > 0$. This together with $(a.)$ and $(b.)$ imply that $g(r)$ has a unique minimum at some point $r_0 > 1$. Finally we show that $r_0 < 2$. For this just calculate $\frac{d}{dr}\, g(2) = 4\lambda\, 8 \int_0^{\frac{\pi}{2}} \frac{4 - 4sin^2\theta}{(4 - sin^2\theta)^{5/2}}\, d\theta = 32\lambda \int_0^{\frac{\pi}{2}} \frac{4cos^2\theta}{(4 - sin^2\theta)^{5/2}}\, d\theta > 0$. ■

Note that, by definition of g, r_0 does not depend on λ even though $g(r)$ does. From the Lemma above we can have an idea how the graph of $g(r)$ looks like:



Define $K_0 = \sqrt{g(r_0)} = \sqrt{r_0^3 \frac{d}{dr} V(r_0)}$. From the Lemma follows that the restriction of $g(r)$ to the interval $(1, r_0]$ is a decreasing function. Also $g(r)$ restricted to the interval $[r_0, \infty)$ is an increasing function. Hence the same is true for the function $\sqrt{g(r)}$. Therefore the functions $\sqrt{g}\,|_{(1,r_0]} : (1, r_0] \to [K_0, \infty)$ and $\sqrt{g}\,|_{[r_0,\infty)} : [r_0, \infty) \to [K_0, \infty)$ have inverses which we call r_1 and r_2 respectively. It is not difficult to verify that (iv), (v), (vii), $(viii)$, (ix) hold for r_1 and r_2. Next we prove (vi). Fix $K > K_0$. Note that $U(r) = \frac{K^2}{2r^2} + V(r) = \frac{K^2}{2r^2} - 4\lambda \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{r^2 - sin^2\theta}} = \frac{1}{r}\Big(\frac{K^2}{2r} - 4\lambda \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1 - \frac{sin^2\theta}{r^2}}}\Big)$. Since $\Big(\frac{K^2}{2r} - 4\lambda \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1 - \frac{sin^2\theta}{r^2}}}\Big)$ tends to $-2\pi\lambda$ when $r \to \infty$, then for large r we have that $U(r)$ is negative. This together with (ii), (iii) of the statement of Theorem

C and the fact that $U(r)$ has exactly two critical points at r_1 and r_2 imply that $\frac{d}{dr}U(r) > 0$ for $1 < r < r_1$ and $r > r_2$. To prove that $\frac{d}{dr}U(r) < 0$ for $r_1 < r < r_2$ it is enough to prove that $\frac{d}{dr}U(r_0) < 0$. But $\frac{d}{dr}U(r_0) = -\frac{K^2}{r_0^3} + \frac{d}{dr}V(r_0) = -\frac{K^2}{r_0^3} + \frac{K_0^2}{r_0^3} = -\frac{K^2-K_0^2}{r_0^3} < 0$.

Finally we prove (x). Let $\bar{r}$ denote r_1 or r_2. Since $K^2 = \bar{r}^3\frac{dV}{dr}(\bar{r})$ we have $U(\bar{r}) = \frac{K^2}{2\bar{r}^2} + V(\bar{r}) = \frac{1}{2}\bar{r}\frac{dV}{dr}(\bar{r}) + V(\bar{r}) = \frac{1}{2}\bar{r}(4\lambda\int_0^{\frac{\pi}{2}}\frac{\bar{r}}{(\bar{r}^2-sin^2\theta)^{3/2}}d\theta) - 4\lambda\int_0^{\frac{\pi}{2}}\frac{d\theta}{(\bar{r}^2-sin^2\theta)^{1/2}} = 2\lambda\int_0^{\frac{\pi}{2}}\frac{2sin^2\theta-\bar{r}^2}{(\bar{r}^2-sin^2\theta)^{3/2}}d\theta = 2\lambda\int_0^{\frac{\pi}{2}}\frac{2cos^2\theta-\bar{r}^2}{(\bar{r}^2-cos^2\theta)^{3/2}}d\theta$, where the last equality is obtained by a change of variable.

If $K \to +\infty$, we have (by $(viii)$ and (ix)) that $r_1 \to 1^+$ and $r_2 \to +\infty$. Hence it is enough to prove that $\lim_{r\to 1}\int_0^{\frac{\pi}{2}}\frac{2cos^2\theta-r^2}{(r^2-cos^2\theta)^{3/2}}d\theta = +\infty$ and $\lim_{r\to+\infty}\int_0^{\frac{\pi}{2}}\frac{2cos^2\theta-r^2}{(r^2-cos^2\theta)^{3/2}}d\theta = 0$. The last limit is clearly zero. We prove now $\lim_{r\to 1}\int_0^{\frac{\pi}{2}}\frac{2cos^2\theta-r^2}{(r^2-cos^2\theta)^{3/2}}d\theta = +\infty$. We have $\int_0^{\frac{\pi}{2}}\frac{2cos^2\theta-r^2}{(r^2-cos^2\theta)^{3/2}}d\theta = \int_0^{\frac{\pi}{6}}\frac{2cos^2\theta-r^2}{(r^2-cos^2\theta)^{3/2}}d\theta + \int_{\frac{\pi}{6}}^{\frac{\pi}{2}}\frac{2cos^2\theta-r^2}{(r^2-cos^2\theta)^{3/2}}d\theta$. Since $\lim_{r\to 1}\int_{\frac{\pi}{6}}^{\frac{\pi}{2}}\frac{2cos^2\theta-r^2}{(r^2-cos^2\theta)^{3/2}}d\theta = \int_{\frac{\pi}{6}}^{\frac{\pi}{2}}\frac{2cos^2\theta-1}{sin^3\theta}d\theta$ is finite, it is enough to prove that $\lim_{r\to 1}\int_0^{\frac{\pi}{6}}\frac{2cos^2\theta-r^2}{(r^2-cos^2\theta)^{3/2}}d\theta = \infty$. We can assume $r < \sqrt{\frac{5}{4}}$. Hence $2cos^2\theta - r^2 \geq \frac{1}{4}$, for $\theta \in [0, \frac{\pi}{6}]$. Therefore $\int_0^{\frac{\pi}{6}}\frac{2cos^2\theta-r^2}{(r^2-cos^2\theta)^{3/2}}d\theta \geq \frac{1}{4}\int_0^{\frac{\pi}{6}}\frac{d\theta}{((r^2-1)+sin^2\theta)^{3/2}}$. But since $(r^2-1) < \frac{1}{4}$, $sin^2\theta \leq sin\theta < 1$ and $sin\theta \leq \theta$ we have $(r^2-1) + sin^2\theta < \frac{5}{4}$. Hence $((r^2-1)+sin^2\theta)^{3/2} < \frac{\sqrt{5}}{2}((r^2-1)+sin\theta) \leq \frac{\sqrt{5}}{2}((r^2-1)+\theta)$.

Consequently $\int_0^{\frac{\pi}{6}}\frac{d\theta}{((r^2-1)+sin^2\theta)^{3/2}} \geq \frac{2}{\sqrt{5}}\int_0^{\frac{\pi}{6}}\frac{d\theta}{(r^2-1)+\theta} = \frac{2}{\sqrt{5}}[ln((r^2-1)+\frac{\pi}{6}) - ln(r^2-1)] \to \infty$ as $r \to 1$. ■

Before we prove Proposition 3 we need some definitions and comments. Since $V(\mathbf{r})$, $\mathbf{r} \in \mathbb{R}^3 - \mathcal{C}$, is invariant by rotations around the z-axis we can reduce our problem in a canonical way to a problem with two degrees of freedom. Using cylindrical coordinates (r, φ, z), the Lagrangian in these coordinates can be written as: $L(r,\varphi,z,\dot{r},\dot{\varphi},\dot{z}) = \frac{1}{2}(\dot{r}^2 + r^2\dot{\varphi}^2 + \dot{z}^2) - V(r,z)$, where $V(r,z) = V(\mathbf{r})$, $\mathbf{r} = (rcos\varphi, rsin\varphi, z)$. Then it is straightforward to verify that the system $\ddot{\mathbf{r}} = -\nabla V(\mathbf{r})$ in these coordinates is given by:

$$\begin{cases} \ddot{r} = \dfrac{K^2}{r^3} - \dfrac{\partial V}{\partial r}(r,z) \\ \\ \ddot{z} = -\dfrac{\partial V}{\partial z}(r,z) \\ \\ \dot{\varphi} = \dfrac{K}{r^2} \end{cases} \tag{5.7}$$

where K is the (constant) angular momentum. Note that the first two equations of system (5.7) can be rewritten as $(\ddot{r},\ddot{z}) = -\nabla\overline{U}\Big(r,z\Big)$, with $\overline{U}\Big(r,z\Big) = \frac{K^2}{2r^2} + V(r,0,z)$. If $(r(t),z(t))$ is a solution of the first two equations of (5.7), defining $\varphi(t) = \int_0^t\frac{K\,ds}{r^2(s)}$, we have that $(r(t),\varphi(t),z(t))$ is a solution of (5.7). Then $(r(t)\,cos\varphi(t), r(t)\,sin\varphi(t), z(t))$ is a solution of $\ddot{\mathbf{r}} = -\nabla V(\mathbf{r})$. Note that if $z \equiv 0$, system (5.7) is reduced to (3.4) and $\overline{U}$ becomes the effective potential U.

Proof of Proposition 3. Suppose $\mathbf{s} = (r,z)$ is an equilibrium position of $\ddot{\mathbf{s}} = -\nabla\overline{U}(\mathbf{s})$. Then $\nabla\overline{U}(\mathbf{s}) = 0$. In particular $\frac{\partial V}{\partial z} = 0$. But $\frac{\partial V}{\partial z} = \beta z$, where $\beta = \int_{\mathcal{C}}\frac{\lambda\,du}{\|\mathbf{r}-u\|^3} > 0$. Hence $z = 0$, i.e the solution lies in the horizontal plane. Since $z = 0$, we have that $\overline{U}(r,0) = U(r)$, where U is the effective potential.

Let $(r, 0)$ be an equilibrium position of $\ddot{\mathbf{s}} = -\nabla\overline{U}(\mathbf{s})$. If $1 < r < r_0$ we know that the corresponding circular solution is not stable. To prove that the circular solution that corresponds to $(r, 0)$, $r > r_0$, is stable it is enough to prove that $(r, 0)$ is a strict local minimum of $\overline{U}$.

Since $(r, 0)$ is a critical point of $\overline{U}$, we have that r is a critical point of U, and since $r > r_0$, $r = r_2(|K|)$ where K is the angular momentum of the solution. Hence r is a strict local minimum of $U(r)$ that is $U(r) < U(r')$ for r' close to r, $r' \neq r$. But an easy calculation from the definition of V shows that $V(r', 0, 0) \leq V(r', 0, z)$, for all r', z, $(r', z) \neq (\pm 1, 0)$ and the same holds for $\overline{U}$. This implies that $\overline{U}(r', z) \geq \overline{U}(r', 0) = U(r') > U(r) = \overline{U}(r, 0)$ for r' close to r, $r \neq r'$. Therefore $r = r_2(|K|)$ is a strict local minimum of $\overline{U}$. ∎

To finish this section we prove that the origin is the only equilibrium position. Note that by symmetry it is a simple matter to show that the origin is in fact an equilibrium position.

Proposition 5.2 *The origin is only equilibrium solution of the system* $\ddot{\mathbf{r}} = -\nabla V(\mathbf{r})$.

Proof. For $\mathbf{r} = (x, y, z)$ differentiating we have that $\frac{\partial V}{\partial z}(\mathbf{r}) = \beta z$, where $\beta = \int_{\mathcal{C}} \frac{\lambda\, du}{\|\mathbf{r} - u\|^3} > 0$. Then if $\nabla V(\mathbf{r}) = 0$ then $z = 0$. Hence every equilibrium position lies in the xy-plane. The Proposition now follows from (i) of Theorem B and (i) of Theorem C. ∎

6 Proof of Proposition 4.

Fix $\lambda > 0$. Consider the fixed homogeneous circle $\mathcal{C}_\epsilon$ with constant density λ, contained in the xy-plane and *passing through the origin*, with radius $\frac{1}{\epsilon}$ and center $(\frac{1}{\epsilon}, 0, 0)$. Note that the xz-plane is an invariant subspace of our problem. Denote the potential of this translated fixed homogeneous circle, and restricted to the xz-plane, by $W(x, z; \epsilon)$. For notational purposes in what follows we use coordinates (x, y) instead of coordinates (x, z).

This potential can be written in the form $W(x, y; \epsilon) = -4\lambda \frac{1}{\epsilon} T(D, d)$ where T, D, d are as in section 1.1. If $x \leq \frac{1}{\epsilon}$ we have that $D^2 = (\frac{2}{\epsilon} - x)^2 + y^2$ and $d^2 = x^2 + y^2$. Hence we have (see section 1.1) $W(x, y; \epsilon) = -4\lambda \frac{c}{2} f\left(\frac{x^2+y^2}{(c-x)^2+y^2}\right) / \sqrt{(c-x)^2 + y^2}$, where $c = \frac{2}{\epsilon}$.

Let ∇W denote the gradient $(\frac{\partial W}{\partial x}, \frac{\partial W}{\partial y})$ of W. A straightforward calculation shows that for $x \leq c = \frac{2}{\epsilon}$ we have

Lemma 6.1

$$\nabla W(x, y; \epsilon) = 2\lambda c \frac{f(t)}{D^3}(x - c, y) - 4\lambda \frac{c^2}{D^5} f'(t)\,[c(x, y) + (y^2 - x^2, -2xy)],$$

where $t = \frac{x^2+y^2}{(c-x)^2+y^2}$ *and* $D = \sqrt{(c-x)^2 + y^2}$.

Remark. If $z = x + iy$ then $z^2 = (x^2 - y^2, 2xy)$, hence the identity above becomes

$$\nabla W\left(z; \tfrac{c}{2}\right) = 2\lambda \tfrac{c}{|z-c|^3} f\left(\tfrac{|z|^2}{|z-c|^2}\right)(z-c) - 4\lambda \tfrac{c^2}{|z-c|^5} f'\left(\tfrac{|z|^2}{|z-c|^2}\right)(cz - z^2) =$$

$$= -2\lambda \tfrac{c^2}{|z-c|^3} f\left(\tfrac{|z|^2}{|z-c|^2}\right) + \left[2\lambda \tfrac{c}{|z-c|^3} f\left(\tfrac{|z|^2}{|z-c|^2}\right) - 4\lambda \tfrac{c^3}{|z-c|^5} f'\left(\tfrac{|z|^2}{|z-c|^2}\right)\right] z + 4\lambda \tfrac{c^2}{|z-c|^5} f'\left(\tfrac{|z|^2}{|z-c|^2}\right) z^2$$

$$= -2\lambda \tfrac{c}{|z-c|^3} \left\{ f\left(\tfrac{|z|^2}{|z-c|^2}\right) + z\left(\tfrac{2c}{|z-c|^2}\right) f'\left(\tfrac{|z|^2}{|z-c|^2}\right)\right\}(z-c).$$

Note that the gradient vector has components in the z and z^2 directions and in the real $(1,0)$ direction.

Now set $h_1(x,y;\epsilon) = \dfrac{4\lambda}{\epsilon} \dfrac{f(t)}{D^3} \dfrac{1}{\epsilon^{3/2}}$, $h_2(x,y;\epsilon) = -\dfrac{32\lambda}{\epsilon^3} \dfrac{1}{D^5} \dfrac{1}{\epsilon^2}$ and $h_3(x,y;\epsilon) = \dfrac{16\lambda}{\epsilon^2} \dfrac{f'(t)}{D^5} \dfrac{1}{\sqrt{\epsilon}}$. Hence

$$\nabla W(x,y;\epsilon) = [\epsilon^{\frac{3}{2}} h_1](x,y) - [2\sqrt{\epsilon}\, h_1](1,0) + [\epsilon^2 f'(t) h_2](x,y) + [\sqrt{\epsilon}\, h_3](x^2 - y^2, 2xy).$$

The next Lemma shows that h_1, h_2, h_3 are bounded in certain sense.

Lemma 6.2 *Given a compact $C \subset I\!R^2 - \{0\}$ there are $\epsilon_0 > 0$, $K > 0$ such that $|h_i(x,y;\epsilon)| < K$ for all $(x,y) \in C$, $0 < \epsilon \le \epsilon_0$.*

Proof. Given a compact $C \subset \mathbb{R}^2 - \{0\}$, there exist constants C_1, C_2, such that $0 < C_1 \le \|(x,y)\| = \sqrt{x^2+y^2} \le C_2$, for all $(x,y) \in C$. For $\epsilon_0 < \frac{1}{C_2}$, we have $C_2 < \frac{1}{\epsilon_0}$, which implies that $C \subset \{(x,y); x < \frac{1}{\epsilon}, y \in \mathbb{R}\}$, for all $\epsilon \le \epsilon_0$, and $h_i(x,y;\epsilon)$ is defined in C, for $\epsilon \le \epsilon_0$, $i = 1,2,3$.

Since $\sqrt{(2-\epsilon x)^2 + \epsilon^2 y^2} = \|(2,0) - \epsilon\,(x,y)\| \le \|(2,0)\| + \epsilon\,\|(x,y)\| = 2 + \epsilon\,\|(x,y)\|$ and $\|(2,0) - \epsilon\,(x,y)\| \ge \|(2,0)\| - \epsilon\,\|(x,y)\| = 2 - \epsilon\,\|(x,y)\|$, and we obtain

$$2 - \epsilon\, C_2 \le \sqrt{(2-\epsilon x)^2 + \epsilon^2 y^2} \le 2 + \epsilon\, C_2 \tag{6.8}$$

Therefore, using Lemma 1.2, part (iii), and (6.8) we have that:

$$|h_1(x,y;\epsilon)| = \left|\frac{4\lambda}{\epsilon} \frac{f(t)}{D^3} \frac{1}{\epsilon^{3/2}}\right| = \frac{4\lambda\sqrt{\epsilon}\, f(t)}{((2-\epsilon x)^2 + \epsilon^2 y^2)^{3/2}} \le \frac{4\lambda\sqrt{\epsilon}\, f(t)}{(2-\epsilon C_2)^3} \le \frac{2\pi\lambda\sqrt{\epsilon}}{(2-\epsilon C_2)^3 \sqrt[4]{t}} =$$
$$= \frac{2\pi\lambda\sqrt{\epsilon}\,[(2-\epsilon x)^2 + \epsilon^2 y^2]^{1/4}}{(2-\epsilon C_2)^3 \sqrt{\epsilon}(x^2+y^2)^{1/4}} \le \frac{2\pi\lambda(2+\epsilon C_2)^{1/2}}{(2-\epsilon C_2)^3 C_1^{1/2}} \le \frac{2\pi\lambda(2+\epsilon_0 C_2)^{1/2}}{(2-\epsilon_0 C_2)^3 C_1^{1/2}} = K_1(C,\epsilon_0).$$

This proves the Lemma for h_1.

Now, applying (6.8) we have $|h_2(x,y;\epsilon)| = \frac{32\lambda}{D^5 \epsilon^5} = \frac{32\lambda}{((2-\epsilon x)^2 + \epsilon^2 y^2)^{5/2}} \le \frac{32\lambda}{(2-\epsilon C_2)^5} \le \frac{32\lambda}{(2-\epsilon_0 C_2)^5} = K_2(C,\epsilon_0)$, and this proves the Lemma for h_2.

Finally, using Lemma 1.2, part (v) and (6.8), we have $|h_3(x,y;\epsilon)| = \left|\frac{16\lambda}{\epsilon^2} \frac{f'(t)}{D^5} \frac{1}{\sqrt{\epsilon}}\right| = \frac{16\lambda}{\epsilon^{5/2}} \frac{|f'(t)|}{\left[\frac{(2-\epsilon x)^2 + \epsilon^2 y^2}{\epsilon^2}\right]^{5/2}}$

$$= \frac{16\lambda\epsilon^{5/2}|f'(t)|}{((2-\epsilon x)^2+\epsilon^2 y^2)^{5/2}} \le \frac{4\lambda\pi}{(x^2+y^2)^{5/4}} \frac{1}{((2-\epsilon x)^2+\epsilon^2 y^2)^{5/4}} \le \frac{4\lambda\pi}{C_1^{5/2}} \frac{1}{(2-\epsilon C_2)^{5/2}} \le \frac{4\lambda\pi}{C_1^{5/2}} \frac{1}{(2-\epsilon_0 C_2)^{5/2}} = K_3(C,\epsilon_0). \blacksquare$$

Write $h(x,y;\epsilon) = (x^2+y^2)\epsilon^2 f'(t) h_2$. Then $\nabla W = [\epsilon^{\frac{3}{2}} h_1](x,y) - [2\sqrt{\epsilon}\, h_1](1,0) + h\, \frac{(x,y)}{(x^2+y^2)} + [\sqrt{\epsilon}\, h_3](x^2 - y^2, 2xy)$.

Lemma 6.3 *If* $(x_n, y_n, \epsilon_n) \to (x, y, 0) \neq (0,0,0)$*, with* $\epsilon_n \neq 0$*,* $(x_n, y_n) \notin \mathcal{C}_{\epsilon_n}$*, then* $lim_{n\to+\infty} h(x_n, y_n; \epsilon_n) = 64\lambda$.

Proof. First note that $\lim_{n\to+\infty} h_2(x_n, y_n; \epsilon_n) = -32\lambda$. Also $\lim_{n\to+\infty} \frac{\epsilon_n^2}{t_n} = \frac{4}{x^2+y^2}$, where $t_n = \frac{x_n^2+y_n^2}{\left(x_n - \frac{2}{\epsilon_n}\right)^2 + y_n^2}$. Therefore $\lim_{n\to+\infty} h(x_n, y_n, \epsilon_n) = \lim_{n\to+\infty}(x_n^2+y_n^2)(\frac{\epsilon_n^2}{t_n})(f'(t_n)t_n)h_2(x_n, y_n; \epsilon_n) = (x^2+y^2)\frac{4}{x^2+y^2}(-\frac{1}{2})(-32\lambda) = 64\lambda$, where we are using the fact that $\lim_{t\to 0^+} tf(t) = -\frac{1}{2}$ (see Lemma 1.2). ■

Define $\nabla W(x, y; 0) := 64\,\lambda\frac{(x,y)}{x^2+y^2}$, $(x, y) \neq (0, 0)$. Hence $\nabla W(x, y; \epsilon)$ is defined on $A = \{\,(x, y; \epsilon)\,;\, x^2 + y^2 \neq 0,\, (x, y) \neq (\frac{2}{\epsilon}, 0)\,\}$, and for each ϵ, ∇W is analytic.

Proof of Proposition 4. Clearly ∇W is continuous in (x, y, ϵ), with $\epsilon \neq 0$. Let $\{\,(x_n, y_n, \epsilon_n)\,\}$ be a sequence in A, with $(x_n, y_n, \epsilon_n) \to (x, y, 0)$, $(x, y) \neq (0, 0)$, $\epsilon_n \neq 0$. Thus $C = \{\,(x_n, y_n),\, n \in \mathbb{N}\,\} \cup \{\,(x, y)\,\}$ is compact and $(0, 0) \notin C$. By Lemma 6.2 there exist n_0 and $K_C > 0$, such that $|h_i((x_n, y_n, \epsilon_n)| < K_C$, for $n \geq n_0$, $i = 1, 2, 3$. Therefore, $\lim_{n\to+\infty} \nabla W(x_n, y_n, \epsilon_n) = \lim_{n\to+\infty} h(x_n, y_n; \epsilon_n)\frac{(x_n,y_n)}{x_n^2+y_n^2} = 64\lambda\frac{(x,y)}{x^2+y^2}$. ■

7 Proof of Proposition 5.

Take $M = 1$. Let $\beta(t) = (D(t), d(t))$ with $\beta(0) = (D, d)$, $\dot\beta(0) = (u, v)$. Let us calculate $\sigma(t) = \sigma(D(t), d(t))$. The following change of variables will simplify our calculations. Write $u = \gamma D$ and $v = (\gamma + \delta)d$. The Taylor series of $D(t), d(t)$ around $t = 0$ are:

$$D(t) = D + tu + \ldots = (1 + t\gamma)D + O(t^2)$$
$$d(t) = d + tv + \ldots = (1 + t\gamma)d + t\delta d + O(t^2)$$

Lemma 7.1 $\sigma(t) = (1+t\alpha^*)\sigma + O(t^2)$*, with* $\alpha^* = \alpha_0 + \delta\sum_{n=1}^{+\infty}\frac{1}{2^n}\frac{d_{n-1}}{D_n}\prod_{j=1}^{n-1}\left(1 - \frac{d_{j-1}}{D_j}\right)$*,* $\alpha_0 = \gamma$*,* $d_0 = d$*,* $D_0 = D$*,* $D_n = \frac{D_{n-1}(0)+d_{n-1}(0)}{2}$*,* $d_n = \sqrt{D_{n-1}(0)d_{n-1}(0)}$*, for* $n \geq 1$*.*

Proof. Calculating the arithmetic-geometric mean of $D(t)$ and $d(t)$, we have:

$$D_1(t) = \frac{D(t) + d(t)}{2} = (1 + t\gamma)D_1 + \frac{1}{2}t\delta d + O(t^2) = (1 + t\alpha_1)D_1 + O(t^2),$$
$$d_1(t) = \sqrt{D(t)d(t)} = (1 + t\gamma)d_1 + \frac{1}{2}t\delta d_1 + O(t^2) = (1 + t\alpha_1)d_1 + \frac{1}{2}t\delta d_1\left(1 - \frac{d}{D_1}\right) + O(t^2),$$

where $\alpha_1 = \gamma + \frac{1}{2}\delta\frac{d}{D_1}$, $D_1 = \frac{D(0)+d(0)}{2} = \frac{D+d}{2}$ and $d_1 = \sqrt{D(0)d(0)} = \sqrt{Dd}$.

Calculating the arithmetic and geometric mean successively, rearranging and defining $\alpha_0 = \gamma$, $\alpha_1 = \gamma + \frac{1}{2}\delta\frac{d}{D_1}$ and

$$\alpha_n = \alpha_{n-1} + \frac{1}{2^n}\delta\frac{d_{n-1}}{D_n}\prod_{j=1}^{n-1}\left(1 - \frac{d_{j-1}}{D_j}\right),\ \text{for}\ \ n \geq 2$$

we obtain, by induction, the general rule:

$$D_n(t) = (1 + t\alpha_n)D_n + O(t^2),$$

$$d_n(t) = (1 + t\alpha_n)d_n + \frac{1}{2^n} t\delta d_n \prod_{j=1}^{n} \left(1 - \frac{d_{j-1}}{D_j}\right) + O(t^2),$$

where $d_0 = d$, $D_0 = D$, $D_n = \frac{D_{n-1}(0)+d_{n-1}(0)}{2}$, $d_n = \sqrt{D_{n-1}(0)d_{n-1}(0)}$, $n \geq 1$.
Note that $\alpha_n = \alpha_0 + \delta \sum_{i=1}^{n} \frac{1}{2^i} \frac{d_{i-1}}{D_i} \prod_{j=1}^{i-1} \left(1 - \frac{d_{j-1}}{D_j}\right)$.

Finally, taking the limit on $d_n(t)$ or on $D_n(t)$, when n goes to infinity, we have

$$\sigma(t) = (1 + t\alpha^*)\sigma + O(t^2),$$

with $\alpha^* = \lim_{n\to\infty} \alpha_n = \alpha_0 + \delta \sum_{n=1}^{+\infty} \frac{1}{2^n} \frac{d_{n-1}}{D_n} \prod_{j=1}^{n-1} \left(1 - \frac{d_{j-1}}{D_j}\right)$. It is easy to see that this series converges. ■

Proposition 5. *We have the following formulas:*

$$\frac{\partial}{\partial D} V(D, d) = \frac{\chi - 1}{D} V(D, d), \qquad \frac{\partial}{\partial d} V(D, d) = -\frac{\chi}{d} V(D, d),$$

where $\chi = \sum_{n=1}^{+\infty} \frac{1}{2^n} \frac{d_{n-1}}{D_n} \prod_{j=1}^{n-1} \left(1 - \frac{d_{j-1}}{D_j}\right)$.

Proof. Since $V(P(t)) = \frac{1}{\sigma(t)} = \frac{1}{\sigma} - \frac{t\alpha^*}{\sigma} + O(t^2) = V(D, d) - t\alpha^* V(D, d) + O(t^2)$ we can now calculate the partial derivaties $\frac{\partial}{\partial d} V(D, d)$, $\frac{\partial}{\partial D} V(D, d)$.

We have $\frac{d}{dt} V(\beta(t))_{|t=0} = -\alpha^* V(D, d) = \langle\, (\frac{\partial}{\partial D} V(\beta(0)), \frac{\partial}{\partial d} V(\beta(0))), (u, v)\,\rangle$. Note that $\alpha^* = \gamma + \delta\chi$, with $\chi = \sum_{n=1}^{+\infty} \frac{1}{2^n} \frac{d_{n-1}}{D_n} \prod_{j=1}^{n-1} \left(1 - \frac{d_{j-1}}{D_j}\right)$, $\gamma = \frac{u}{D}$ and $\delta = \frac{v}{d} - \frac{u}{D}$. We have $\alpha^* = \frac{u}{D} + (\frac{v}{d} - \frac{u}{D})\,\chi = \frac{u}{D}(1-\chi) + \frac{v}{d}\chi$. Therefore $\alpha^* = \left(\frac{1-\chi}{D}\right) u + \frac{\chi}{d} v$, so $\langle\, (\frac{\partial}{\partial D} V(D, d), \frac{\partial}{\partial d} V(D, d)), (u, v)\,\rangle = -\left(\frac{1-\chi}{D} V(D, d)u + \frac{\chi}{d} V(D, d)v\right)$ for all u, v. Hence:

$$(\frac{\partial}{\partial D} V(D, d), \frac{\partial}{\partial d} V(D, d)) = \left(\frac{\chi - 1}{D} V(D, d),\ -\frac{\chi}{d} V(D, d)\right). ■$$

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